Logarithmic Functions

Definition. Let a be a positive real number such that $a \neq 1$. The **logarithmic function** with base a is defined by

$$f(x) = \log_a(x),$$

for all $x > 0$. This function is the inverse of the exponential function a^x. In other words, for any $x > 0$ and real number y, we have

$$y = \log_a(x) \quad \Longleftrightarrow \quad a^y = x.$$

Evaluating Logarithmic Expressions

Example. Evaluate the expression $\log_2(8)$.

$$\log_2(8) = y \quad \Longleftrightarrow \quad 2^y = 8$$

$$2^y = 2^3$$

$$y = 3$$

$$\boxed{\log_2(8) = 3}$$

Example. Evaluate the following expressions.

Problem	Answer
$\log_3(81)$	4
$\log_{10}(1000)$	3
$\log_5(\frac{1}{5})$	-1
$\log_4(2)$	1/2

$3^y = 81$
$3^y = 3^4$
$y = 4$

$10^y = 1000$
$10^y = 10^3$
$y = 3$

$5^y = \frac{1}{5}$
$5^y = 5^{-1}$
$y = -1$

$4^y = 2$
$(2^2)^y = 2$
$2^{2y} = 2^1$
$2y = 1$
$y = 1/2$

Graphing Logarithmic Functions

The graph of the logarithmic function $f(x) = \log_a(x)$ (with $a > 1$) has several key features that help us understand its behavior:

- **Domain:**

  $(0, \infty)$. $\log_a(x) = y$ means that $a^y = x$. If $a>0$ then $a^y>0$. So $x>0$.

- **Range:**

  $(-\infty, \infty)$. We can raise a to any real number. So $\log_a(x)$ can be anything

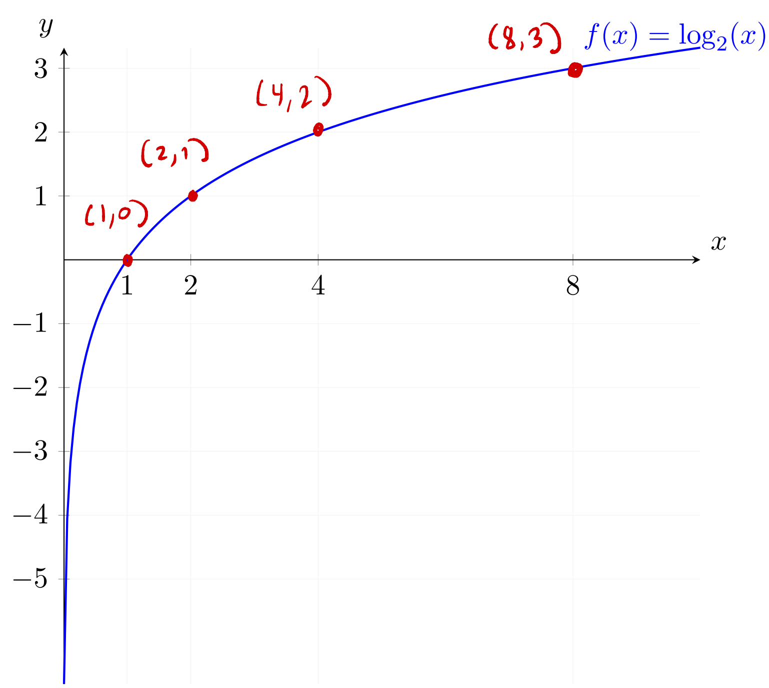
- **Vertical Asymptote:**

  The line $x=0$ is an asymptote. As $x \to 0$ from the right, we evaluate things like $\log_a(0.00001)$. This means $a^y = 0.00001$ and y is a large neg. number.

- **End Behavior:**

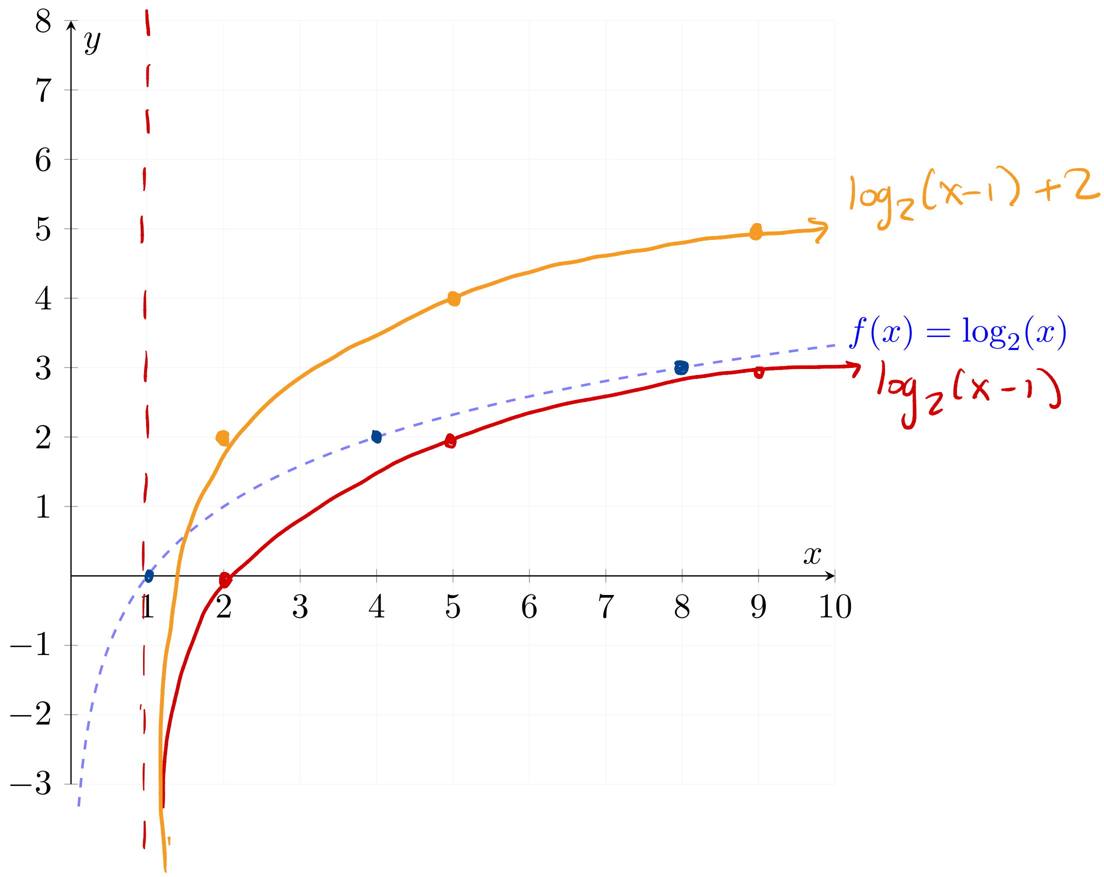
  As $x \to \infty$, $\log_a(x) \to \infty$

Below is an example graph of the function $f(x) = \log_2(x)$ to illustrate these features:



Graph Transformations

Example. Graph the function $g(x) = \log_2(x-1) + 2$.



- shift right 1
- shift up 2

Common and Natural Logarithms

Definition.

- The **common logarithm** is the logarithm with base 10:

$$\log(x) = \log_{10}(x).$$

- The **natural logarithm** is the logarithm with base e (where $e \approx 2.71828$):

$$\ln(x) = \log_e(x).$$

Finding Domains of Logarithmic Functions

Example. Determine the domain of the function $f(x) = \ln(4-x^2)$, express your answer in interval notation.

We need $4-x^2 > 0$

$\Rightarrow \quad x^2 < 4$

$\Rightarrow \quad -2 < x < 2$

$\boxed{(-2, 2)}$

Example. Determine the domain of the function $g(x) = \log_3(2x+1)$, and express your answer in interval notation.

We need $2x+1 > 0$

$2x > -1$

$x > -\frac{1}{2}$

$\boxed{\left(-\frac{1}{2}, \infty\right)}$

Inverses of Exponential and Logarithmic Functions

Exponential and logarithmic functions are inverses of each other. In general,

$$f(x) = a^x \quad \iff \quad f^{-1}(x) = \log_a(x).$$

Example. Find the inverse of $f(x) = 3^x$.

$y = 3^x$

$\log_3(y) = x$ (by def of log)

$$\boxed{f^{-1}(x) = \log_3(x)}$$

Example. Find the inverse of $g(x) = \log_5(x)$.

$y = \log_5(x)$

$5^y = x$ (by def of log)

$$\boxed{g^{-1}(x) = 5^x}$$