Exponential Functions

Definition of the Exponential Function

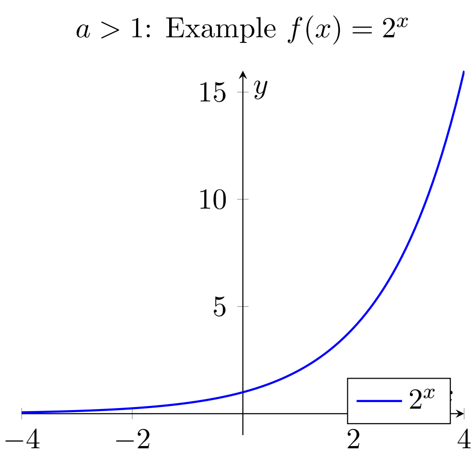
Definition. Let a be a positive real number with $a \neq 1$. The *exponential function* with base a is defined by

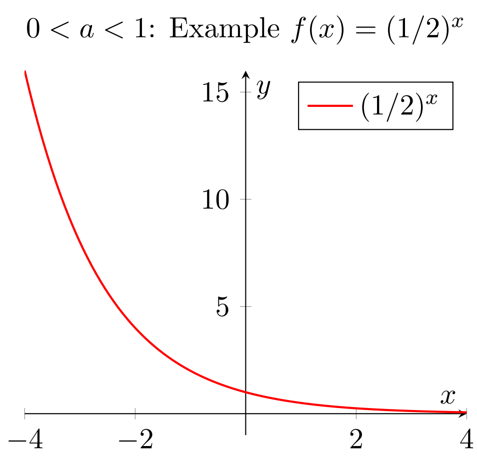
Graphs and Transformations of Exponential Functions

The exponential function $f(x) = a^x$ has the following properties:

- **Domain:**
- **Range:**
- **End Behavior:**

The following graphs illustrate these cases:





Transformations: Shifts, reflections, and stretches can be applied to a^x to model different scenarios.

Modeling Situations Using Exponential Functions

Exponential models often take the form

$$f(x) = C \cdot a^x,$$

or, when incorporating concepts like doubling time or half-life,

$$f(t) = C \cdot 2^{t/t_0} \quad \text{or} \quad f(t) = C \cdot \left(\frac{1}{2}\right)^{t/t_0},$$

where C is the initial amount and t_0 represents the doubling time (or half-life).

Example. A bacteria culture doubles every 3 hours. If the initial population is 200, determine the population after 9 hours.

Example. A radioactive substance has a mass of 50 grams at $t = 0$ and 25 grams at $t = 4$ hours. Model the decay using the form

$$M(t) = C \cdot a^t,$$

and determine C and a.

Defining the Number e

Definition. The number e is defined as the unique real number satisfying

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828.$$

It serves as the base for the natural exponential function e^x.

Modeling Compound Interest

Compound interest can be modeled in two different ways: periodically and continuously.

Periodic Compounding

If an initial amount P is compounded n times per year at an annual interest rate r, the amount after t years is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Continuous Compounding

When interest is compounded continuously, the amount after t years is:

$$A = Pe^{rt}.$$

Example. An investment of \$1000 is made at an annual interest rate of 5%, compounded monthly. Determine the amount after 10 years.

Example. Now, consider the same investment compounded continuously. Find the amount after 10 years.