Exponential Functions

Definition of the Exponential Function

Definition. Let a be a positive real number with $a \neq 1$. The *exponential function* with base a is defined by

$$f(x) = a^x$$

Graphs and Transformations of Exponential Functions

The exponential function $f(x) = a^x$ has the following properties:

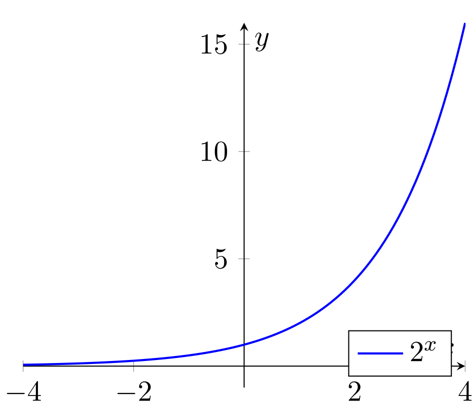
- **Domain:** $(-\infty, \infty)$
- **Range:** $(0, \infty)$
- **End Behavior:**

If $a > 1$: as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 0$

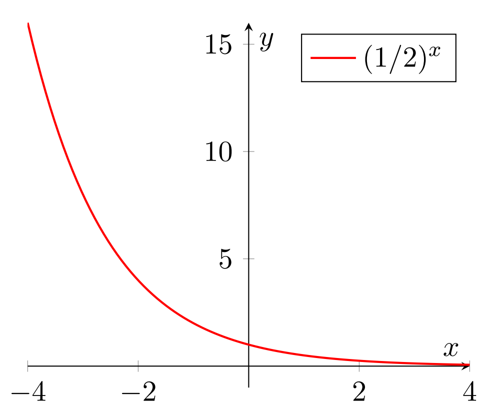
If $a < 1$: as $x \to \infty$, $f(x) \to 0$
as $x \to -\infty$, $f(x) \to \infty$

The following graphs illustrate these cases:

$a > 1$: Example $f(x) = 2^x$



$0 < a < 1$: Example $f(x) = (1/2)^x$



Transformations: Shifts, reflections, and stretches can be applied to a^x to model different scenarios.

Modeling Situations Using Exponential Functions

Exponential models often take the form

$$f(x) = C \cdot a^x,$$

or, when incorporating concepts like doubling time or half-life,

$$f(t) = C \cdot 2^{t/t_0} \quad \text{or} \quad f(t) = C \cdot \left(\frac{1}{2}\right)^{t/t_0},$$

where C is the initial amount and t_0 represents the doubling time (or half-life).

Example. A bacteria culture doubles every 3 hours. If the initial population is 200, determine the population after 9 hours.

how many times you double

$$P(t) = 200 \cdot (2)^{t/3}$$

$$P(9) = 200 \cdot 2^{9/3}$$

$$= 200 \cdot 2^3$$

$$= \underbrace{200}_{\text{initial}} \cdot \underbrace{2 \cdot 2 \cdot 2}_{\text{we double 3 times in 9 hours}}$$

$$= 1600$$

Example. A radioactive substance has a mass of 50 grams at $t = 0$ and 25 grams at $t = 4$ hours. Model the decay using the form

$$M(t) = C \cdot a^t,$$

and determine C and a.

At $t=0$: $M(0) = C \cdot a^0 = C \cdot 1 = C \Rightarrow C = 50$

At $t=4$: $M(4) = 50 \cdot a^4 = 25$

$\Rightarrow 50 \cdot a^4 = 25$

$\Rightarrow a^4 = \frac{1}{2}$

$\Rightarrow a = \sqrt[4]{\frac{1}{2}}$

Hence $M(t) = 50 \cdot \left(\sqrt[4]{1/2}\right)^t$ $\left(\text{or } M(t) = 50 \cdot \left(\frac{1}{2}\right)^{t/4}\right)$

Defining the Number e

Definition. The number e is defined as the unique real number satisfying

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828\ldots\ldots$$

It serves as the base for the natural exponential function e^x.

Modeling Compound Interest

Compound interest can be modeled in two different ways: periodically and continuously.

Periodic Compounding

If an initial amount P is compounded n times per year at an annual interest rate r, the amount after t years is given by:

$$A = P\left(1+\frac{r}{n}\right)^{nt}.$$

Continuous Compounding

When interest is compounded continuously, the amount after t years is:

$$A = Pe^{rt}.$$

Example. An investment of \$1000 is made at an annual interest rate of 5%, compounded monthly. Determine the amount after 10 years.

$P = 1000 \quad r = 0.05 \quad n = 12 \quad t = 10$

$$A = 1000\left(1+\frac{0.05}{12}\right)^{12\cdot 10}$$

$$= \$1647.01$$

Example. Now, consider the same investment compounded continuously. Find the amount after 10 years.

$$A = Pe^{rt}$$

$$A = 1000 \cdot e^{0.05 \cdot 10}$$

$$A \approx \$1648.72$$