Rational Functions

Definition of a Rational Function

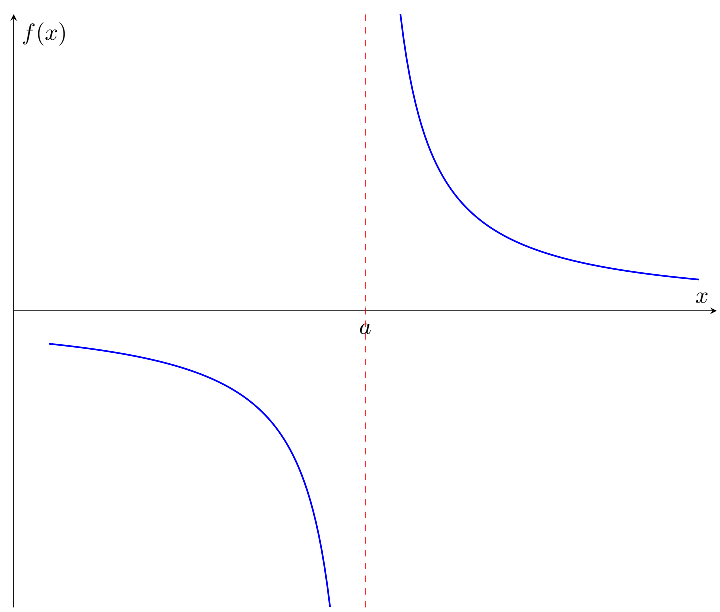
Definition. A *rational function* is any function that can be written as

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The domain of f consists of all real numbers x for which $q(x) \neq 0$.

Vertical Asymptotes and Infinite Behavior

A vertical asymptote occurs at $x = a$ when the function grows without bound as $x \to a$.



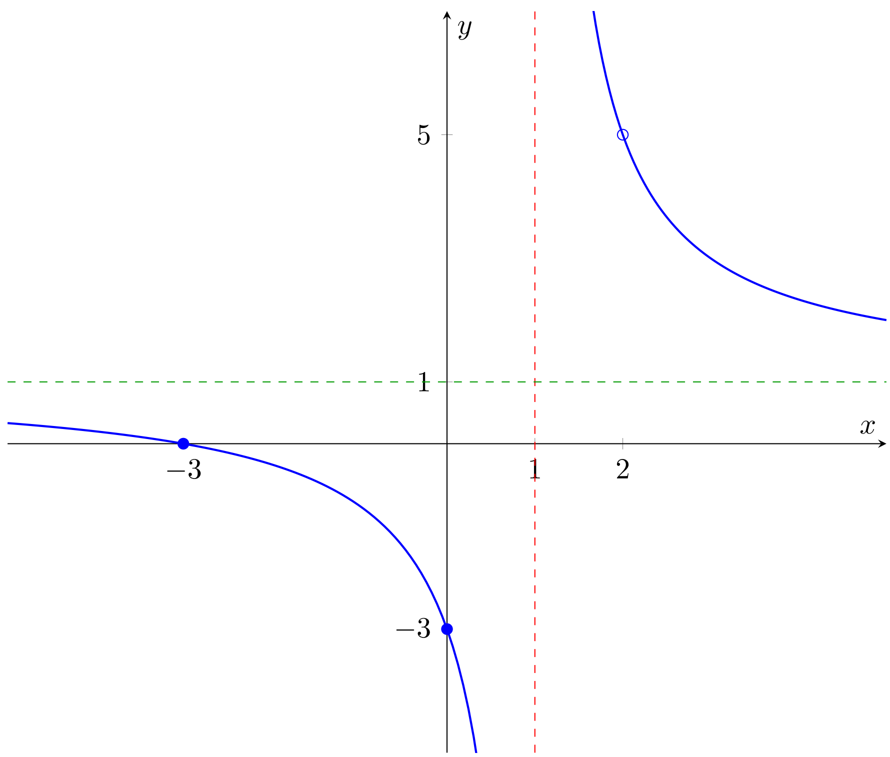
Analyzing a Rational Function

Given a rational function, we determine its domain, intercepts, vertical asymptotes, holes, and horizontal asymptote. We then use this information to sketch its graph.

Example. Consider the rational function

$$f(x) = \frac{x^2 + x - 6}{x^2 - 3x + 2} = \frac{(x-2)(x+3)}{(x-2)(x-1)}.$$

The following graph summarizes our findings:



When finding vertical asymptotes analytically, factor both the numerator and denominator. Cancel any common factors:

- A cancelled factor indicates a removable discontinuity (a hole).
- Any factor remaining in the denominator indicates a vertical asymptote at the corresponding value.

Finding Horizontal Asymptotes: End Behavior

Horizontal asymptotes describe the end behavior of a rational function. We can find them using two methods:

Example. Find the horizontal asymptote of

$$f(x) = \frac{2x^2 + 3x + 1}{x^2 - 5}.$$

Slant Asymptotes via Polynomial Long Division

When the degree of the numerator is exactly one more than that of the denominator, the rational function has a *slant* (oblique) asymptote. We use polynomial long division to find it.

Example. Find the slant asymptote of

$$f(x) = \frac{x^2 + x + 1}{x - 1}.$$