Rational Functions

Definition of a Rational Function

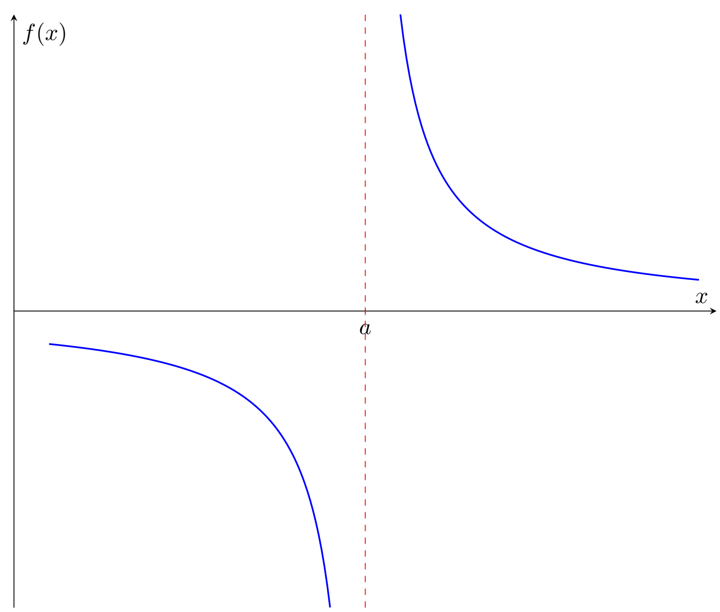
Definition. A *rational function* is any function that can be written as

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The domain of f consists of all real numbers x for which $q(x) \neq 0$.

Vertical Asymptotes and Infinite Behavior

A vertical asymptote occurs at $x = a$ when the function grows without bound as $x \to a$.



Calculus notation: $\lim\limits_{x \to a^-} f(x) = -\infty$, $\lim\limits_{x \to a^+} f(x) = \infty$

As x goes to a from the left, $f(x)$ goes to $-\infty$

As x goes to a from the right, $f(x)$ goes to ∞

Analyzing a Rational Function

Given a rational function, we determine its domain, intercepts, vertical asymptotes, holes, and horizontal asymptote. We then use this information to sketch its graph.

Example. Consider the rational function

$$f(x) = \frac{x^2 + x - 6}{x^2 - 3x + 2} = \frac{(x-2)(x+3)}{(x-2)(x-1)}.$$

Domain: The denominator must be nonzero

$$(x-2)(x-1) \neq 0$$

$$x \neq 2 \quad \text{and} \quad x \neq 1$$

Holes: The factor $(x-2)$ cancels in both the numerator and the denominator. Hence $x=2$ is a hole. To find the y-coordinate, simplify the function

$$f(x) = \frac{x+3}{x-1}, \quad x \neq 1 \text{ and } x \neq 2$$

Substitute $x = 2$

$$f(2) = \frac{5}{1} = 5$$

The hole occurs at $(2, 5)$

Vertical Asymptote: After cancelation, there is a remaining factor of $x-1$ in the denominator. Hence $x=1$ is a vertical asymptote.

Intercepts:

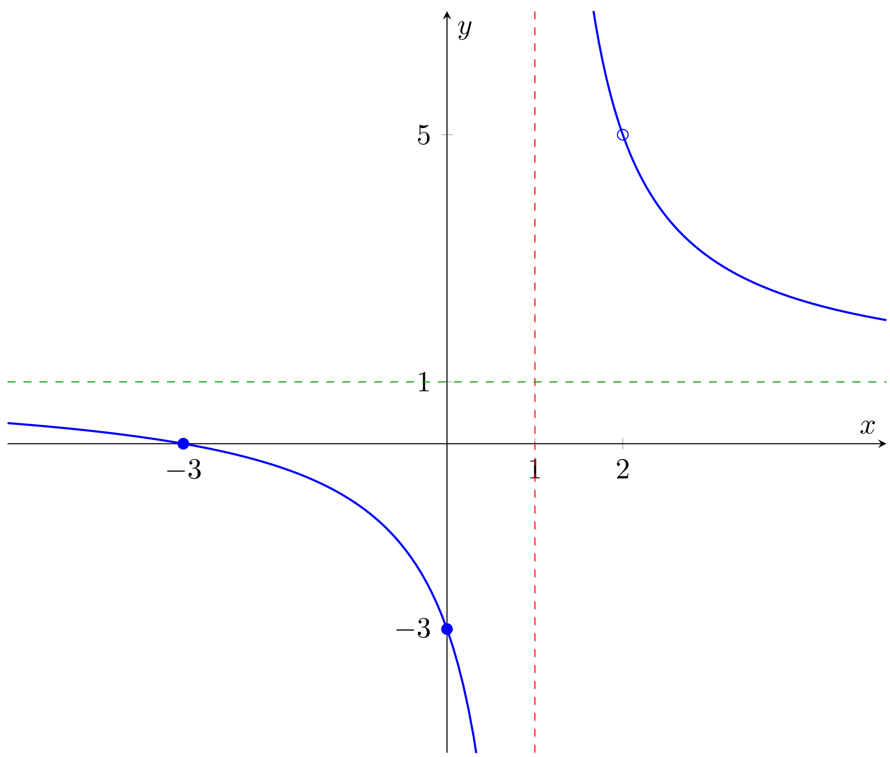
x-intercepts: Set the simplified numerator to 0: $x+3=0 \Rightarrow x=-3$

$(-3, 0)$

y-intercept: Compute $f(0) = \frac{0+3}{0-1} = -3 \Rightarrow (0, -3)$

Horizontal Asymptote: Because the numerator and denominator have the same degree, this is given by the ratio of the leading coefficients: $y = \frac{1}{1} = 1$

The following graph summarizes our findings:



When finding vertical asymptotes analytically, factor both the numerator and denominator. Cancel any common factors:

- A cancelled factor indicates a removable discontinuity (a hole).
- Any factor remaining in the denominator indicates a vertical asymptote at the corresponding value.

Finding Horizontal Asymptotes: End Behavior

Horizontal asymptotes describe the end behavior of a rational function. We can find them using two methods:

Example. Find the horizontal asymptote of

$$f(x) = \frac{2x^2 + 3x + 1}{x^2 - 5}.$$

Method 1: Factor out the highest power of x

$$f(x) = \frac{x^2\left(2 + \cancel{\frac{3}{x}}^{\,0} + \cancel{\frac{1}{x^2}}^{\,0}\right)}{x^2\left(1 - \cancel{\frac{5}{x^2}}^{\,0}\right)}$$

As $x \to \pm\infty$, the terms $\frac{3}{x}$, $\frac{1}{x^2}$, and $\frac{5}{x^2}$ all go to 0

As $x \to \pm\infty$, $f(x) \approx \frac{2x^2}{x^2} = 2$

This gives a horizontal asymptote at $y = 2$

Method 2: If the degrees are the same, the horizontal asymptote is given by the ratio of the leading coefficients.

$$y = \frac{2}{1} = 2$$

Slant Asymptotes via Polynomial Long Division

When the degree of the numerator is exactly one more than that of the denominator, the rational function has a *slant* (oblique) asymptote. We use polynomial long division to find it.

Example. Find the slant asymptote of

$$f(x) = \frac{x^2 + x + 1}{x - 1}.$$

$$\begin{array}{r|l} & x + 2 \\ \hline x - 1 & x^2 + x + 1 \\ & x^2 - x \\ \hline & \quad 2x + 1 \\ & \quad 2x - 2 \\ \hline & \qquad 3 \end{array}$$

Hence we can express $\dfrac{x^2+x+1}{x-1} = x+2 + \dfrac{3}{x-1}$

As $x \to \pm\infty$, $\dfrac{3}{x-1} \to 0$ so we have a slant asymptote at $y = x+2$