Graphing Polynomials and the Fundamental Theorem of Algebra

Introduction

Polynomials are one of the most fundamental objects in mathematics, appearing in algebra, calculus, and applied sciences. In this lecture, we will explore the properties and behaviors of polynomial functions, focusing on their definitions, roots, and graphical representations.

Defining Polynomials

A polynomial function is defined as a function of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where:

- $a_n, a_{n-1}, \ldots, a_0$ are constants (coefficients),
- n is a non-negative integer (the degree of the polynomial),
- $a_n \neq 0$ (the leading coefficient).

Example. Describe the polynomial $P(x) = 2x^3 - 5x^2 + x - 7$.

P(x) is a degree 3 polynomial with leading coefficient 2

The Fundamental Theorem of Algebra

Theorem (The Fundamental Theorem of Algebra)**.** Every non-zero polynomial of degree n with complex coefficients has exactly n roots in the complex number system (counting multiplicities).

Question. What does the Fundamental Theorem of Algebra imply?

A degree n polynomial always has n solutions, although some may be complex or repeated.

Example. Find the roots of $P(x) = x^3 - 4x$.

GCF

A difference of squares

Factor the polynomial: $P(x) = x(x^2-4) = x(x-2)(x+2)$

The roots are $x=0$, $x=2$, and $x=-2$

Behavior of Polynomials at Zeros and Multiplicities

The behavior of a polynomial near its zeros is determined by the multiplicity of each zero. Here is how to compute and interpret multiplicity.

Definition. What is the multiplicity of a zero $x = c$?

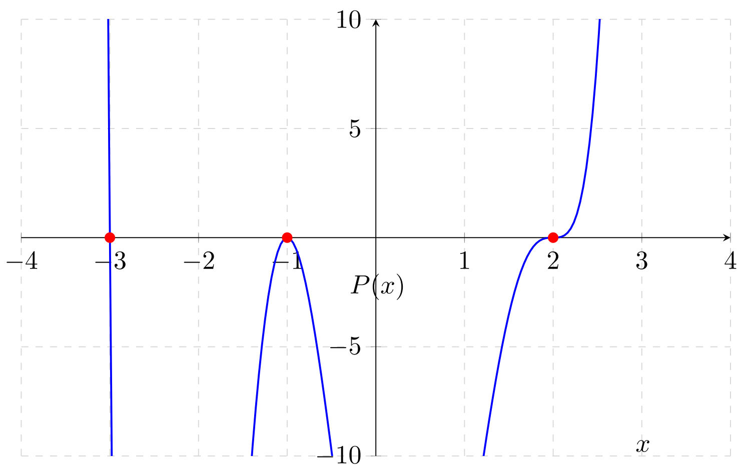
It is the exponent k of the corresponding factor $(x-c)^k$ in the polynomial. In other words, how many times $(x-c)$ appears in the factorization.

Question. How to interpret multiplicity graphically?

Odd Multiplicity: The graph cross the x-axis at $x = c$

Even Multiplicity: The graph touches the x-axis at $x=c$, but does not cross it.

Example. Analyze the polynomial $P(x) = (x+1)^2(x-2)^3(x+3)$.



Zero	Multiplicity	Behavior
$x=-1$	2 (even)	Doesn't cross at $x=-1$
$x = 2$	3 (odd)	Crosses at $x=2$
$x = -3$	1 (odd)	Crosses at $x=-3$

End Behavior of Polynomials

The end behavior of a polynomial is determined by its leading term (the term with the highest power of x).

Example. Determine the end behavior of $P(x) = 3x^4 - 5x^3 + x - 7$ using the formal method.

- Factor out the highest power of x

$$P(x) = x^4\left(3 - \cancel{\frac{5}{x}}^{\,0} - \cancel{\frac{1}{x^3}}^{\,0} - \cancel{\frac{7}{x^4}}^{\,0}\right)$$

- As $x \to \infty$, $\frac{5}{x}$, $\frac{1}{x^3}$, and $\frac{7}{x^4}$ all go to 0.

  Hence $P(x) \approx 3x^4$. So $P(x) \to \infty$ as $x \to \infty$.

- As $x \to -\infty$, $\frac{5}{x}$, $\frac{1}{x^3}$, and $\frac{7}{x^4}$ all go to 0

  $P(x) \to \infty$ as $x \to -\infty$ as well

Example. Determine the end behavior of $P(x) = -2x^3 + 5x^2 - x + 1$ using the leading term.

The leading term is $-2x^3$

As $x \to \infty$, $-2x^3 \to -\infty$. So $P(x) \to -\infty$

As $x \to -\infty$, $-2x^3 \to \infty$. So $P(x) \to \infty$

Graphing Factored Polynomials with a Sign Chart

A number line sign chart helps determine the sign of $P(x)$ between its zeros.

Example. Graph $P(x) = (x-1)(x+2)^2$ using a sign chart.

1. Identify zeros and their multiplicities

	Multiplicity	Behavior
$x = 1$	1 (odd)	Crosses
$x = -2$	2 (even)	Touches (doesn't cross)

2. Create a sign chart

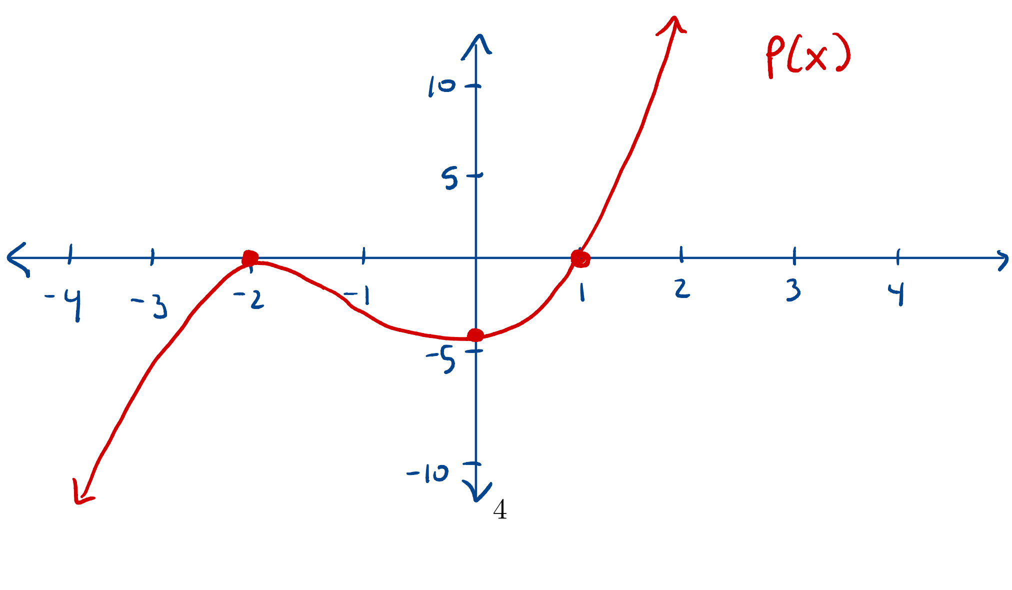
Interval	$(x-1)$	$(x+2)^2$	$P(x)$
$(-\infty, -2)$	−	+	−
$(-2, 1)$	−	+	−
$(1, \infty)$	+	+	+

3. Key Points:

zeros: $(-2,0)$ and $(1,0)$

additional points: $(0,-4)$, $(2,16)$

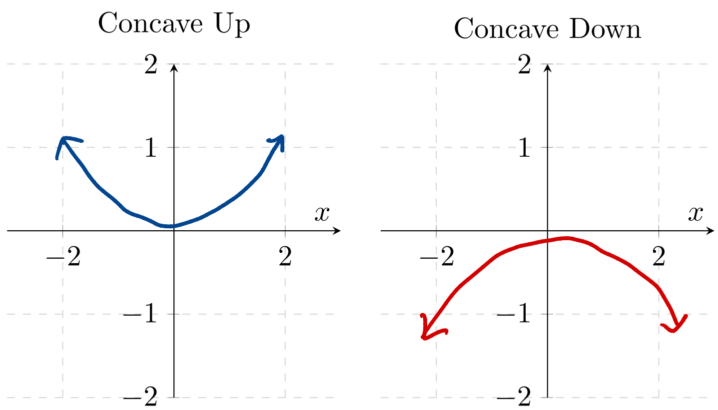
4.



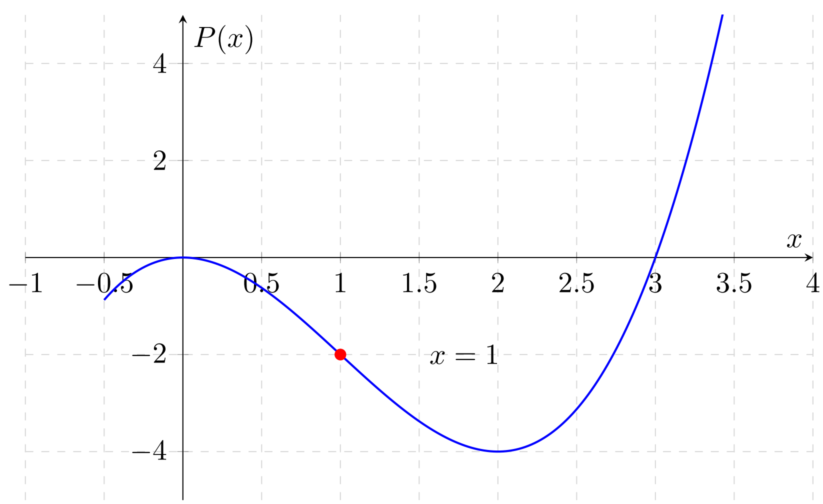
Understanding Concavity in Polynomials

Concavity describes how a curve bends:

- **Concave up:** The graph looks like a cup holding water.
- **Concave down:** The graph looks like a cup spilling water.



Example. Analyze the concavity of $P(x) = x^3 - 3x^2$.



$(-\infty, 1)$ concave down

$(1, \infty)$ concave up