Solving Quadratic Equations

Introduction

Quadratic equations appear frequently in mathematics, science, and engineering. In this lecture, we will explore multiple techniques to solve quadratic equations.

Solving Quadratic Equations by Factoring

Factoring involves rewriting a quadratic equation in the form $(x - r_1)(x - r_2) = 0$, where r_1 and r_2 are the solutions.

Example. Solve $x^2 - 5x + 6 = 0$ by factoring.

This factors as $(x-2)(x-3) = 0$. Set each factor to 0

$x - 2 = 0 \Rightarrow x = 2$

$x - 3 = 0 \Rightarrow x = 3$

The solutions are $x = 2$ or $x = 3$

Solving Quadratic Equations by Factoring by Grouping

When a quadratic equation cannot be easily factored, factoring by grouping can be used.

Example. Solve $2x^2 + 5x + 3 = 0$ by factoring by grouping.

Rewrite this as $2x^2 + 2x + 3x + 3 = 0$

Group terms $(2x^2 + 2x) + (3x + 3) = 0$

Factor: $2x(x+1) + 3(x+1) = 0$

Factor again: $(2x+3)(x+1) = 0$

2x ☺ + 3 ☺ = (2x+3) ☺

Set each factor to 0:

$2x + 3 = 0 \Rightarrow x = \frac{-3}{2}$

$x + 1 = 0 \Rightarrow x = -1$

Solutions are $x = -\frac{3}{2}$ and $x = -1$

Solving Quadratic Equations Using the Quadratic Formula

The quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provides a way to solve any quadratic equation.

Example. Solve $x^2 - 4x - 5 = 0$ using the quadratic formula.

Here, $a = 1$, $b = -4$, and $c = -5$

$$(x-5)(x+1)$$

$$x = 5 \text{ or } x = -1$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1}$$

$$x = \frac{4 \pm \sqrt{16 + 20}}{2 \cdot 1}$$

$$x = \frac{4 \pm \sqrt{36}}{2}$$

$$x = \frac{4 \pm 6}{2} \Rightarrow x = \frac{4+6}{2} \text{ or } x = \frac{4-6}{2}$$

$$\boxed{x = 5 \quad \text{or} \quad x = -1}$$

Solving Quadratic Equations of the Form $a(x-h)^2 + k = 0$

Equations of this form can be solved by isolating the squared term and taking the square root of both sides.

Example. Solve $2(x-3)^2 + 8 = 0$.

$$2(x-3)^2 + 8 = 0$$

isolate squared term: $2(x-3)^2 = -8$

$$(x-3)^2 = -4$$

No solution since $(x-3)^2 \geq 0$ for all x

Instead, if this were:

$$2(x-3)^2 - 8 = 0$$

$$2(x-3)^2 = 8$$

$$(x-3)^2 = 4$$

$$x - 3 = 2 \text{ or } x - 3 = -2$$

$$\boxed{x = 5 \quad \text{or} \quad x = 1}$$

Solving Non-Quadratic Equations Quadratic in Structure Using Substitution

Equations of higher degrees or with complex terms can often be rewritten as quadratic equations by substituting $u = x^k$.

Example. Solve $x^4 - 5x^2 + 4 = 0$.

Let $u = x^2$. The equation becomes $u^2 - 5u + 4 = 0$

Factor : $(u-4)(u-1) = 0$

$u = 4$ or $u = 1$

Back-substitute : $x^2 = 4$ or $x^2 = 1$

$x = \pm 2$ or $x = \pm 1$

Completing the Square

Completing the square involves rewriting the quadratic equation in vertex form $a(x-h)^2 + k = 0$.

Example. Solve $x^2 + 6x + 5 = 0$ by completing the square.

Rewrite this as $x^2 + 6x = -5$

> Complete the square

$x^2 + 6x + 9 = -5 + 9$

> Factor

$(x+3)^2 = 4$

> Take square roots

$x + 3 = 2$ or $x + 3 = -2$

> Solve

$x = -1$ or $x = -5$