Circles

Introduction

Circles are a fundamental concept in mathematics, with applications in geometry, physics, engineering, and beyond. In this lecture, we will explore the properties and equations of circles, focusing on their standard form and how to derive key information such as center, radius, and intercepts. We will also learn how to rewrite the general form of a circle's equation into standard form using completing the square and apply distance and midpoint formulas to analyze circles in practical contexts.

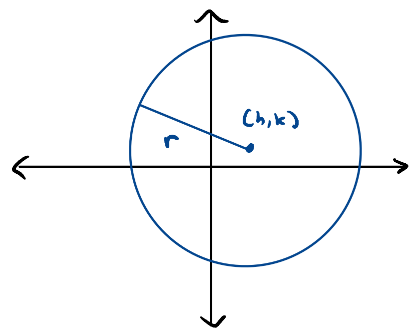
Standard Form of a Circle

Definition. What is the standard form of an equation for a circle?

The standard form for a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

- (h,k) is the center of the circle
- r is the radius of the circle



Example. Write the standard form equation of a circle with center $(4, 0)$ and radius 5.

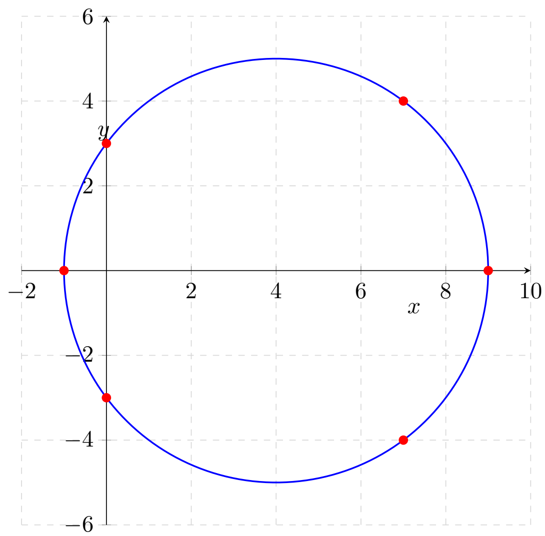
Here, $h=4$ and $k=0$. Also $r=5$

$$(x-4)^2 + (y-0)^2 = 5^2$$

$$\boxed{(x-4)^2 + y^2 = 25}$$

Example. For the circle $(x-4)^2 + y^2 = 25$:

- What are the values of y when $x = 7$?
- What are the values of x when $y = 3$?
- What are the x- and y-intercepts?



- Substitute $x = 7$: $(7-4)^2 + y^2 = 25$

$$3^2 + y^2 = 25$$

$$y^2 = 16 \quad \Rightarrow \quad \boxed{y = \pm 4}$$

- Substitute $y = 3$: $(x-4)^2 + 3^2 = 25$

$$(x-4)^2 = 16$$

$$x - 4 = \pm 4$$

$$x - 4 = -4 \text{ or } x - 4 = 4 \quad \Rightarrow \quad \boxed{x = 0 \text{ or } x = 8}$$

- x-intercepts occur when $y = 0$

$$(x-4)^2 + 0^2 = 25$$

$$(x-4)^2 = 25$$

$$x - 4 = -5 \text{ or } x - 4 = 5$$

$$x = -1 \text{ or } x = 9$$

$$\boxed{(-1, 0) \text{ OR } (9, 0)}$$

y-intercepts occur when $x = 0$

$$(0-4)^2 + y^2 = 25$$

$$16 + y^2 = 25$$

$$y^2 = 9$$

$$y = -3 \text{ or } y = 3$$

$$\boxed{(0, -3) \text{ or } (0, 3)}$$

Converting General Form to Standard Form

Definition. What is the general form of a circle?

General form is

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

* Here, $A = B$ and it is usually 1

Example. Convert the equation $x^2 + y^2 - 6x + 4y + 9 = 0$ to standard form and find the center and radius.

1. Group x- and y- variables

$$(x^2 - 6x) + (y^2 + 4y) = -9$$

2. Complete the square for each group

$$[(x^2 - 6x + 9) - 9] + [(y^2 + 4y + 4) - 4] = -9$$

$$(x-3)^2 - 9 + (y+2)^2 - 4 = -9$$

3. We obtain $(x-3)^2 + (y+2)^2 = 4$

Center is $(3, -2)$ and radius is 2

Compare this to the Equation of a circle

Using Distance and Midpoint Formulas

- The distance formula is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Gives the distance between (x_1, y_1) and (x_2, y_2)

- The midpoint formula is:

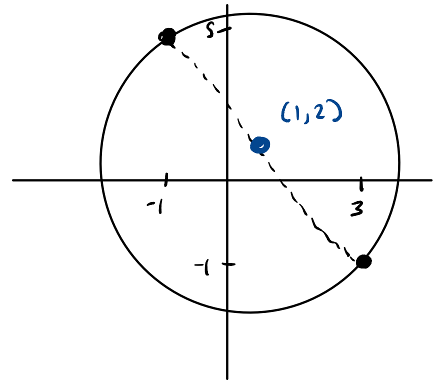
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Gives the midpoint between (x_1, y_1) and (x_2, y_2)

Example. Find the standard form equation of a circle with diameter endpoints $(3, -1)$ and $(-1, 5)$.

1. $$M = \left(\frac{3 + (-1)}{2}, \frac{-1 + 5}{2}\right)$$

$$= (1, 2)$$



2. Find the radius. The diameter is

$$d = \sqrt{(3 - (-1))^2 + ((-1) - 5)^2}$$

$$= \sqrt{(4)^2 + (-6)^2}$$

$$= \sqrt{16 + 36}$$

$$= \sqrt{52} = 2\sqrt{13} \Rightarrow \boxed{\text{Radius: } \sqrt{13}}$$

3. $$(x - 1)^2 + (y - 2)^2 = (\sqrt{13})^2$$

$$\boxed{(x-1)^2 + (y-2)^2 = 13}$$

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Application: Communication Networks

Circles are often used to model coverage areas for wireless communication networks, such as Wi-Fi routers or cellular towers. The center of the circle represents the location of the communication device, and the radius represents its maximum range.

Example. A cellular tower is located at the point $(5, 3)$ and has a maximum coverage range of 10 kilometers. The *boundary* of the coverage area is modeled by the equation:

$$(x - 5)^2 + (y - 3)^2 = 100.$$

- Determine whether a user's device at $(12, 3)$ is within the coverage area of the tower.
- Find the x-values for which $y = 3$ on the boundary of the coverage area.

Example. Another cellular tower is located at $(-3, 0)$ with a coverage radius of 7 kilometers.

- Write the equation modeling the coverage boundary of this second tower.
- If a user is located at $(1, 0)$, determine if they are covered by at least one of the towers.

Example. Two cellular towers are located at $(3, 1)$ and $(10, 4)$. Each has a coverage radius of 6 kilometers.

- Write the equations modeling the coverage boundaries of these two towers.
- Determine the distance between the two towers.
- **Challenge:** Do their coverage areas overlap? Justify your answer.