— SOLUTIONS —

Midterm 2

Differential Geometry of Curves and Surfaces
MATH 4230/5230
Fall 2026

Friday, October 30, 2026

PRACTICE EXAM

NAME: _____

1 problem; 100 points total.

Problem 1: 100 points.

Problem:	1	Total
Possible:	100	100
Earned:		

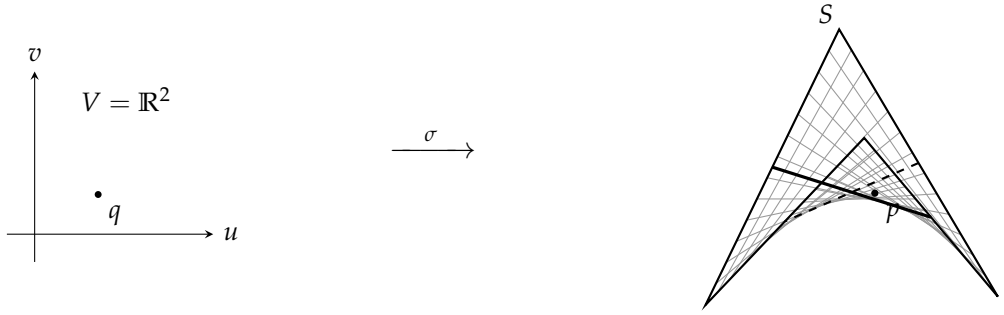
- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete the exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with this **cover sheet** and the questions in the correct order.
- You have 45 minutes to complete the exam.

Problem 1 **100 points**

Let $S \subseteq \mathbb{R}^3$ be a regular oriented surface with coordinate patch

$$\sigma(u, v) = (u, v, uv), \qquad (u, v) \in V = \mathbb{R}^2.$$

We will use the orientation of S determined by σ. Let $q = (u, v)$ and $p = \sigma(q)$.



$$q = (u, v) \qquad \longmapsto \qquad p = \sigma(q)$$

You may use without proof that the Jacobian matrix is given by

$$J(\sigma) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v & u \end{bmatrix}.$$

Problem 1, continued.

1(a) **(15 points)** *Use the cross product to derive the following formula for the Gauss map*

$$N : S \longrightarrow S^2 \subseteq \mathbb{R}^3$$

in the given local coordinates:

$$N(\sigma(u,v)) = \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}.$$

Recall that $\sigma(u,v) = (u,v,uv)$, and we are using the orientation given by σ.

Solution.

The columns of the Jacobian matrix are denoted σ_u and σ_v:

$$J(\sigma) = \begin{bmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v & u \end{bmatrix}.$$

A compatible normal vector is therefore

$$\sigma_u \times \sigma_v = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & v \\ 0 & 1 & u \end{bmatrix} = (-v,-u,1).$$

Its length is

$$|\sigma_u \times \sigma_v| = \sqrt{1+u^2+v^2}.$$

Consequently,

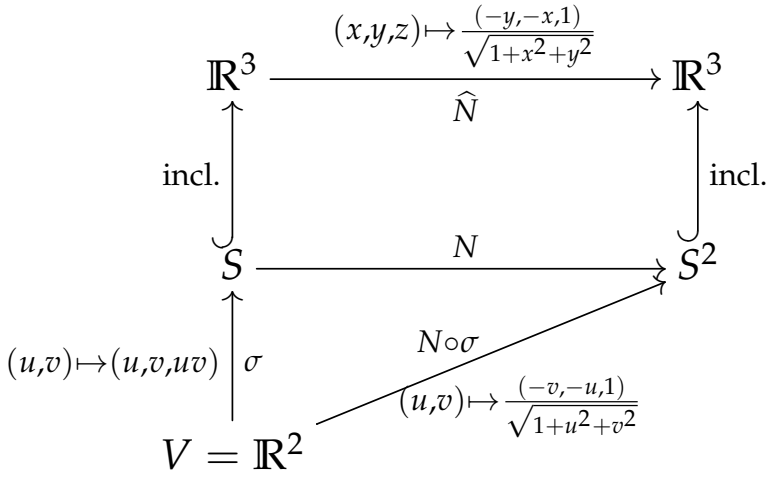
$$\boxed{N(\sigma(u,v)) = \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} = \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}.}$$

Problem 1, continued.

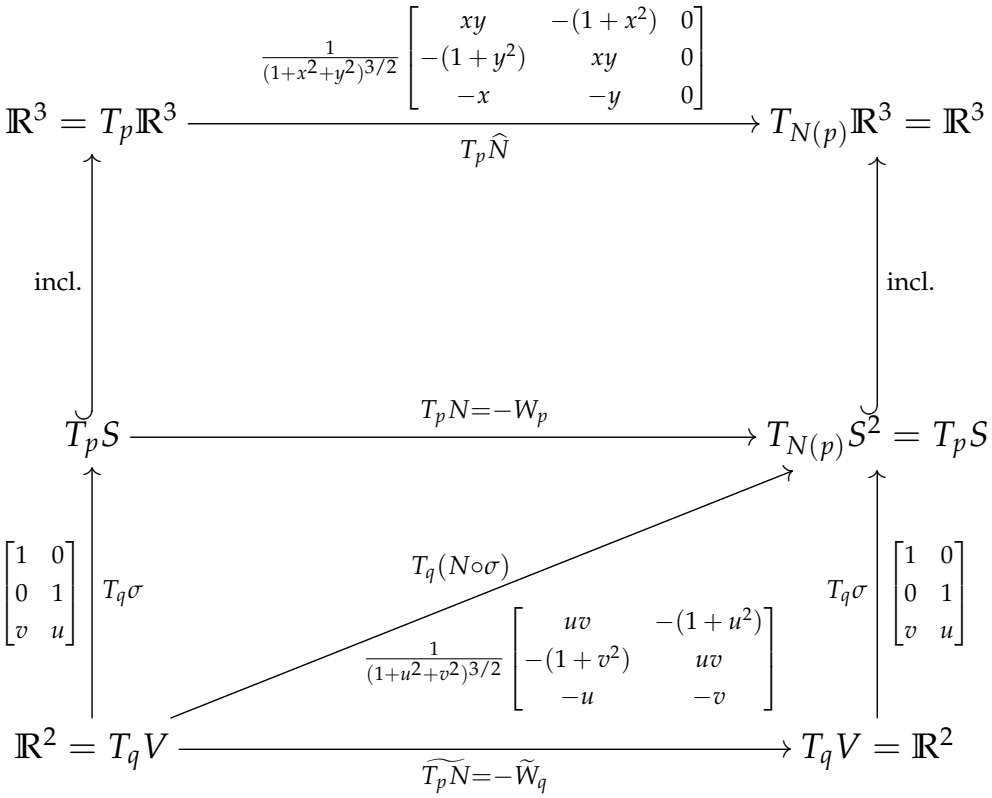
Define an ambient extension $\widehat{N}$ to $\mathbb{R}^3$ of the Gauss map N by

$$\widehat{N} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad \widehat{N}(x,y,z) = \frac{(-y,-x,1)}{\sqrt{1+x^2+y^2}}.$$

We then have the following commutative diagram:



Taking differentials in the previous diagram gives the commutative diagram



Supplementary Explanation

It is not required for the exam, but just for clarity, here is the computation of the Jacobian matrices for $\widehat{N}$ and $N \circ \sigma$. This explains the given diagram; it is not part of a graded solution.

The ambient differential. The product rule applied to $\widehat{N}(x,y,z) = \dfrac{(-y,-x,1)}{\sqrt{1+x^2+y^2}}$ gives

$$\frac{\partial \widehat{N}}{\partial x} = \frac{(0,-1,0)}{\sqrt{1+x^2+y^2}} - \frac{x(-y,-x,1)}{(1+x^2+y^2)^{3/2}},$$
$$\frac{\partial \widehat{N}}{\partial y} = \frac{(-1,0,0)}{\sqrt{1+x^2+y^2}} - \frac{y(-y,-x,1)}{(1+x^2+y^2)^{3/2}}, \qquad \frac{\partial \widehat{N}}{\partial z} = (0,0,0).$$

Here we just did the computation in vector form to make it shorter. We could just as easily have done each coordinate function on its own. Simplifying these three columns gives the 3×3 Jacobian

$$J(\widehat{N}) = \frac{1}{(1+x^2+y^2)^{3/2}} \begin{bmatrix} xy & -(1+x^2) & 0 \\ -(1+y^2) & xy & 0 \\ -x & -y & 0 \end{bmatrix}.$$

The coordinate differential. A similar computation for $N \circ \sigma = \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}$ gives

$$J(N \circ \sigma) = \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} uv & -(1+u^2) \\ -(1+v^2) & uv \\ -u & -v \end{bmatrix}.$$

For instance, the first entry of the matrix is given by

$$\frac{\partial}{\partial u}\frac{-v}{\sqrt{1+u^2+v^2}} = \frac{0}{\sqrt{1+u^2+v^2}} + (-v)\frac{-1}{2}(1+u^2+v^2)^{-3/2}(2u)$$
$$= \frac{uv}{(1+u^2+v^2)^{3/2}}$$

The second entry of the matrix is given by

$$\frac{\partial}{\partial v}\frac{-v}{\sqrt{1+u^2+v^2}} = \frac{-1}{\sqrt{1+u^2+v^2}} + (-v)\frac{-1}{2}(1+u^2+v^2)^{-3/2}(2v)$$
$$= -\frac{1+u^2+v^2}{(1+u^2+v^2)^{3/2}} + \frac{v^2}{(1+u^2+v^2)^{3/2}} = \frac{-(1+u^2)}{(1+u^2+v^2)^{3/2}}$$

The rest of the computations are similar.

Problem 1, continued.

In the next few parts, you will compute the Weingarten map as a matrix in local coordinates.

1(b) **(25 points)** *Compute* $\widetilde{W}_q \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

[Hint: follow $(1,0)$ through $T_q\sigma$ and $T_p\widehat{N}$, or simply compute the image of $(1,0)$ under $T_q(N \circ \sigma)$. Express the result in the basis (σ_u, σ_v), i.e., the columns of the Jacobian matrix, and then apply the minus sign.]

Solution.

Following the first coordinate vector through the upper maps gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{T_q\sigma} \begin{bmatrix} 1 \\ 0 \\ v \end{bmatrix} \xrightarrow{T_p\widehat{N}} \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} uv \\ -1-v^2 \\ -u \end{bmatrix}.$$

Alternatively, in one step, this is the image of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ under the map $T_q(N \circ \sigma)$.

To express this in the basis σ_u, σ_v, write

$$\frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} uv \\ -1-v^2 \\ -u \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ v \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}.$$

Comparing the first two components yields

$$\alpha = \frac{uv}{(1+u^2+v^2)^{3/2}}, \qquad \beta = \frac{-1-v^2}{(1+u^2+v^2)^{3/2}}.$$

(The third component agrees because $uv \cdot v + (-1-v^2)u = -u$.) Thus

$$\widetilde{T_pN} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} uv \\ -1-v^2 \end{bmatrix}$$

$$\widetilde{W}_q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} -uv \\ 1+v^2 \end{bmatrix}.$$

Problem 1, continued.

1(c) **(20 points)** *Compute* $\widetilde{W}_q \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

For the second coordinate vector,

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{T_q\sigma} \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix} \xrightarrow{T_p\widehat{N}} \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} -1-u^2 \\ uv \\ -v \end{bmatrix}.$$

Alternatively, in one step, this is the image of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under the map $T_q(N \circ \sigma)$.

In the coordinate tangent basis this is

$$\frac{-(1+u^2)}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} 1 \\ 0 \\ v \end{bmatrix} + \frac{uv}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}.$$

Applying the minus sign gives

$$\widetilde{W}_q \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} 1+u^2 \\ -uv \end{bmatrix}.$$

1(d) **(5 points)** *Use the previous parts to compute* $\widetilde{W}_q$.

Solution.

$$\boxed{\widetilde{W}_q = \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{bmatrix}.}$$

Problem 1, continued.

1(e) **(20 points)** *Find the Gaussian curvature $K(p)$ and mean curvature $H(p)$ in local coordinates.* [Hint: Use your coordinate Weingarten matrix $\widetilde{W}_q$]

Solution.

The Gaussian curvature is the determinant:

$$\begin{aligned} K(p) = \det \widetilde{W}_q &= \left(\frac{1}{(1+u^2+v^2)^{3/2}}\right)^2 \left(u^2v^2 - (1+u^2)(1+v^2)\right) \\ &= \frac{u^2v^2 - (1+u^2+v^2+u^2v^2)}{(1+u^2+v^2)^3} \\ &= \boxed{-\frac{1}{(1+u^2+v^2)^2}.} \end{aligned}$$

The mean curvature is half the trace:

$$H(p) = \frac{1}{2}\operatorname{tr} \widetilde{W}_q = \frac{1}{2}\frac{-2uv}{(1+u^2+v^2)^{3/2}} = \boxed{-\frac{uv}{(1+u^2+v^2)^{3/2}}}.$$

Problem 1, continued.

1(f) **(15 points)** *What are the principal curvatures at* $p = (1,1,1)$*?*

$$k_1 = _____, \qquad k_2 = _____.$$

[Hint: Consider the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$]

Solution.

At $q = (1,1)$, the coordinate matrix is

$$\widetilde{W}_q = \frac{1}{3\sqrt{3}}\begin{bmatrix}-1 & 2\\ 2 & -1\end{bmatrix}.$$

We have

$$\widetilde{W}_q\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{3\sqrt{3}}\begin{bmatrix}1\\1\end{bmatrix}, \qquad \widetilde{W}_q\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{3\sqrt{3}}\begin{bmatrix}-3\\3\end{bmatrix} = -\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\end{bmatrix}.$$

Thus we may take the following eigenvectors (with associated eigenvalues)

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}, \qquad k_1 = \frac{1}{3\sqrt{3}}, \quad k_2 = -\frac{1}{\sqrt{3}}.$$

The principal curvatures are therefore

$$\boxed{k_1 = \frac{1}{3\sqrt{3}}, \qquad k_2 = -\frac{1}{\sqrt{3}}.}$$

Supplementary Explanation

Although it is not needed for this problem, as a reminder of how to find eigenvectors in general, here is a computation to find the eigenvectors of the matrix $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$.

First we compute the characteristic polynomial

$$p_A(t) = \det(A - tI) = \det \begin{bmatrix} -1-t & 2 \\ 2 & -1-t \end{bmatrix} = t^2 + 2t - 3 = (t-1)(t+3).$$

The real roots of the characteristic polynomial are the eigenvalues; i.e., the eigenvalues are $\lambda = 1, -3$. We then find a basis of each eigenspace; for instance, the eigenspace E_{-3} is defined to be $\ker(A - (-3)I) = \ker \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

To find a basis for this kernel, we put the matrix in reduced row echelon form and then modify:

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

The columns with the red -1s form a basis for the kernel. So a basis for the eigenspace E_{-3} is given by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.