

— SOLUTIONS —

# Midterm 1

Differential Geometry of Curves and Surfaces

MATH 4230/5230

Fall 2026

Wednesday, September 23, 2026

**PRACTICE EXAM**

NAME: \_\_\_\_\_

**4 problems; 25 points each; 100 points total.**

|           |    |    |    |    |       |
|-----------|----|----|----|----|-------|
| Problem:  | 1  | 2  | 3  | 4  | Total |
| Possible: | 25 | 25 | 25 | 25 | 100   |
| Earned:   |    |    |    |    |       |

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete the exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with this **cover sheet**, and the questions in the correct order.
- You have 50 minutes to complete the exam.

**Problem 1****25 points**

Consider the space curve

$$\gamma(t) = \left( t + \frac{t^2}{2} + \frac{t^3}{3}, 1 + t + \frac{t^2}{2} - \frac{t^3}{3}, t^2 \right), \quad -\frac{1}{2} < t < \frac{1}{2}.$$

(a) **(15 points)** Compute the Frenet frame  $\mathbf{t}(0), \mathbf{n}(0), \mathbf{b}(0)$ .

**Solution.** We want to use  $\mathbf{t} = \mathbf{v}/|\mathbf{v}|$ ,  $\mathbf{n} = \mathbf{a}^\perp/|\mathbf{a}^\perp|$ , and  $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ . Differentiating  $\gamma(t)$  gives

$$\gamma'(t) = (1 + t + t^2, 1 + t - t^2, 2t),$$

$$\gamma''(t) = (1 + 2t, 1 - 2t, 2).$$

Therefore, at  $t = 0$ , we have

$$\mathbf{v} = (1, 1, 0) \quad \mathbf{a} = (1, 1, 2).$$

$$|\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \mathbf{a}^\perp = \mathbf{a} - \frac{\langle \mathbf{a}, \mathbf{v} \rangle}{|\mathbf{v}|^2} \mathbf{v} = (1, 1, 2) - \frac{2}{2}(1, 1, 0) = (0, 0, 2).$$

Therefore we have

$$\mathbf{t}(0) = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(1, 1, 0)}{\sqrt{2}},$$

$$\mathbf{n}(0) = \frac{\mathbf{a}^\perp}{|\mathbf{a}^\perp|} = \frac{(0, 0, 2)}{2} = (0, 0, 1),$$

$$\mathbf{b}(0) = \mathbf{t}(0) \times \mathbf{n}(0) = \frac{(1, 1, 0) \times (0, 0, 1)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{(1, -1, 0)}{\sqrt{2}}.$$

**Problem 1, continued.**

For the same curve

$$\gamma(t) = \left( t + \frac{t^2}{2} + \frac{t^3}{3}, 1 + t + \frac{t^2}{2} - \frac{t^3}{3}, t^2 \right),$$

evaluate the following at  $t = 0$ .

- (b) **(5 points)** Compute the curvature  $\kappa(0)$ .

**Solution.** At zero,  $\mathbf{v} = (1, 1, 0)$  and  $\mathbf{a}^\perp = (0, 0, 2)$ . Therefore,

$$\kappa(0) = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} = \frac{2}{1^2 + 1^2} = 1.$$

- (c) **(5 points)** Compute the torsion  $\tau(0)$ .

**[Hint:** For any regular space curve with nonzero curvature, one has  $\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{|\gamma' \times \gamma''|^2}$ .]

**Solution.** Differentiating our formula  $\gamma''(t) = (1 + 2t, 1 - 2t, 2)$  gives

$$\gamma'''(t) = (2, -2, 0).$$

Using our earlier computation of  $\mathbf{v}$  and  $\mathbf{a}$  gives us:

$$\mathbf{v} \times \mathbf{a} = (1, 1, 0) \times (1, 1, 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = (2, -2, 0).$$

Thus  $|\mathbf{v} \times \mathbf{a}|^2 = 2^2 + (-2)^2 = 8$ . The hint gives

$$\tau(0) = \frac{\langle \gamma'(0) \times \gamma''(0), \gamma'''(0) \rangle}{|\gamma'(0) \times \gamma''(0)|^2} = \frac{\langle \mathbf{v} \times \mathbf{a}, \gamma'''(0) \rangle}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{(2)(2) + (-2)(-2)}{8} = 1.$$

**Problem 2****25 points**

Let  $\gamma : I \rightarrow \mathbb{R}^2$  be a unit-speed plane curve, where  $0 \in I$ , and suppose  $\kappa_s(0) \neq 0$ . Let  $M$  be the matrix with columns given by the unit tangent vector  $\gamma'(0)$  and its rotation by  $90^\circ$ :

$$M = \begin{bmatrix} | & | \\ \gamma'(0) & R_{90}\gamma'(0) \\ | & | \end{bmatrix}.$$

(a) **(10 points)** Prove that  $M$  is orthogonal and  $\det M = 1$ .

**Solution.** To show  $M$  is orthogonal, we need to show that  $M^T M = I$ . We have seen that any square matrix whose columns form an orthonormal basis is an orthogonal matrix. Since the vectors  $\gamma'(0)$  and  $R_{90}\gamma'(0)$  form an orthonormal basis in the standard orientation,  $M$  is orthogonal with determinant 1.

We can also prove this with a computation. Suppose  $\gamma'(0) = (a, b)$  with  $a^2 + b^2 = 1$ . Then

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad M^T M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} = I,$$

and  $\det M = a^2 + b^2 = 1$ .

(b) **(5 points)** Define  $f(\mathbf{x}) = M^T(\mathbf{x} - \gamma(0))$  and  $\beta(t) = (f \circ \gamma)(t) = M^T(\gamma(t) - \gamma(0))$ . By part (a),  $f$  is a proper rigid motion, so  $\beta$  is also a unit-speed plane curve. Prove that  $\beta''(0) = (0, \kappa_s(0))$ .

**Solution.** Differentiating  $\beta(t) = (f \circ \gamma)(t) = M^T(\gamma(t) - \gamma(0))$  gives  $\beta'(t) = M^T\gamma'(t)$  and  $\beta''(t) = M^T\gamma''(t)$ . For a unit-speed plane curve,  $\gamma'' = \kappa_s R_{90}\gamma'$  (see Tapp's Equation (1.9)). By construction,  $M^T = M^{-1}$  is the matrix that takes  $R_{90}\gamma'(0)$  to the vector  $(0, 1)$ , so that  $\beta''(0) = M^T\kappa_s(0)R_{90}\gamma'(0) = \kappa_s(0)(0, 1)$ .

Alternatively, we can just carry out the computation. Writing  $\gamma'(0) = (a, b)$  as above,

$$\beta''(0) = \kappa_s(0)M^T R_{90}\gamma'(0) = \kappa_s(0) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} = \kappa_s(0) \begin{pmatrix} 0 \\ a^2 + b^2 \end{pmatrix} = (0, \kappa_s(0)).$$

**Problem 2, continued.**

Recall that  $\gamma$  is unit speed,  $\kappa_s(0) \neq 0$ , and

$$M = \begin{bmatrix} | & | \\ \gamma'(0) & R_{90}\gamma'(0) \\ | & | \end{bmatrix}, \quad \beta(t) = M^T(\gamma(t) - \gamma(0)).$$

You may use part (b):  $\beta''(0) = (0, \kappa_s(0))$ .

(c) **(10 points)** Find the center of the translated osculating circle of  $\beta$  at  $t = 0$ , in terms of  $\kappa_s(0)$ .

[Hint: At a point  $\mathbf{p}$  of a regular curve with curvature  $\kappa > 0$ , the translated osculating circle is the circle with center  $\mathbf{p} + \frac{1}{\kappa}\mathbf{n}$  and radius  $1/\kappa$ . Find  $\beta(0)$ ,  $\beta'(0)$ , and  $\beta''(0)$ .]

**Solution.** We have

$$\begin{aligned} \beta(0) &= M^T(\gamma(0) - \gamma(0)) = 0 \\ \beta'(0) &= M^T\gamma'(0) = M^{-1}\gamma'(0) = (1, 0) = \mathbf{t}_\beta(0) \\ \beta''(0) &= \kappa_s(0)(0, 1) = \kappa_\beta(0)\mathbf{n}_\beta(0) \end{aligned}$$

Consequently, the translated osculating circle is a circle tangent to the  $x$ -axis (since  $\beta'(0) = (1, 0)$ ), with radius  $1/\kappa_\beta(0)$ , and centered therefore at  $(0, 1/\kappa_s(0))$  (above the  $x$ -axis if  $\kappa_s(0)$  is positive, and below the  $x$ -axis if  $\kappa_s(0)$  is negative).

Alternatively, we can argue as follows:

$$\kappa_\beta(0) = |(0, \kappa_s(0))| = |\kappa_s(0)|, \quad \mathbf{n}_\beta(0) = \frac{(0, \kappa_s(0))}{|\kappa_s(0)|}.$$

The hint (Tapp's Equation (1.8), which places the center along the unit normal) now gives

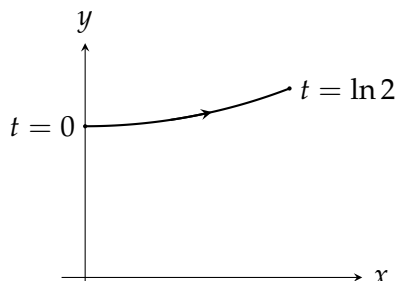
$$0 + \frac{1}{|\kappa_s(0)|} \frac{(0, \kappa_s(0))}{|\kappa_s(0)|} = \left(0, \frac{\kappa_s(0)}{\kappa_s(0)^2}\right) = \left(0, \frac{1}{\kappa_s(0)}\right).$$

For  $\kappa_s(0) < 0$  the center is below the origin; for  $\kappa_s(0) > 0$  it is above.

**Problem 3****25 points**

Consider the plane curve

$$\gamma(t) = \left( t, \frac{e^t + e^{-t}}{2} \right), \quad 0 \leq t \leq \ln 2.$$

(a) **(5 points)** Compute the total arc length  $L$  of this curve.

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**Solution.** Differentiating and simplifying the squared speed gives

$$\gamma'(t) = \left( 1, \frac{e^t - e^{-t}}{2} \right), \quad |\gamma'(t)|^2 = 1 + \frac{e^{2t} - 2 + e^{-2t}}{4} = \left( \frac{e^t + e^{-t}}{2} \right)^2.$$

Thus  $|\gamma'(t)| = (e^t + e^{-t})/2$ , so that

$$L = \int_0^{\ln 2} |\gamma'(t)| dt = \int_0^{\ln 2} \frac{e^t + e^{-t}}{2} dt = \left[ \frac{e^t - e^{-t}}{2} \right]_0^{\ln 2} = \frac{2 - 1/2}{2} - \frac{1 - 1}{2} = \frac{3}{4}.$$

**Problem 3, continued.**

Use the same plane curve

$$\gamma(t) = \left( t, \frac{e^t + e^{-t}}{2} \right), \quad 0 \leq t \leq \ln 2,$$

whose total arc length is denoted by  $L$ .

- (b) **(15 points)** Let  $\tilde{\gamma} : [0, L] \rightarrow \mathbb{R}^2$  be its orientation-preserving unit-speed reparametrization starting at  $\gamma(0)$ . Compute  $\tilde{\gamma}''(L)$ .

**[Hint:** No inverse function is needed; interpret  $\tilde{\gamma}''$  in terms of the curvature, and then compute this from  $\gamma$ .]

**Solution.** For a unit-speed parameterized curve  $\tilde{\gamma}(t)$ , we know that  $\tilde{\gamma}''(t)$  is the curvature vector. Moreover, the curvature vector is unchanged by reparameterization, so we are free to compute the curvature vector for  $\gamma$  at  $t = \ln 2$ .

From the part above, we have  $\gamma'(t) = \left( 1, \frac{e^t - e^{-t}}{2} \right)$ , and then differentiating gives  $\gamma''(t) = \left( 0, \frac{e^t + e^{-t}}{2} \right)$ . At  $t = \ln 2$ , we have  $\mathbf{v} = (1, 3/4)$ ,  $\mathbf{a} = (0, 5/4)$ ,  $|\mathbf{v}|^2 = 25/16$ , and  $\langle \mathbf{a}, \mathbf{v} \rangle = 15/16$  so that

$$\begin{aligned} \vec{\kappa} &= \frac{\mathbf{a}^\perp}{|\mathbf{v}|^2} = \frac{\mathbf{a} - \frac{\langle \mathbf{a}, \mathbf{v} \rangle}{|\mathbf{v}|^2} \mathbf{v}}{|\mathbf{v}|^2} = \frac{(0, \frac{5}{4}) - \frac{\frac{15}{16}}{\frac{25}{16}}(1, \frac{3}{4})}{\frac{25}{16}} = \frac{(0, \frac{5}{4}) - \frac{3}{5}(1, \frac{3}{4})}{\frac{25}{16}} \\ &= \frac{16}{25} \left( -\frac{3}{5}, \frac{4}{5} \right) = \left( -\frac{48}{125}, \frac{64}{125} \right). \end{aligned}$$

- (c) **(5 points)** Let  $\mathbf{F}(x, y) = (y + e^x, x)$ , and let  $C$  be traversed by the given curve from  $t = 0$  to  $t = \ln 2$ . Compute  $\int_C \mathbf{F} \cdot d\gamma$ .

**[Hint:** Look for a potential function.]

**Solution.** For  $f(x, y) = xy + e^x$ , differentiation gives  $\nabla f = (y + e^x, x) = \mathbf{F}$ . We then have (Tapp's Proposition 2.25):

$$\int_C \mathbf{F} \cdot d\gamma = f\left(\ln 2, \frac{5}{4}\right) - f(0, 1) = \left(\frac{5}{4} \ln 2 + 2\right) - 1 = 1 + \frac{5}{4} \ln 2.$$

**Problem 4****25 points**

Let  $\gamma : I \rightarrow \mathbb{R}^3$  be a unit-speed curve with  $\kappa(s) > 0$  and suppose there is a real number  $\lambda$  such that

$$\tau(s) = \lambda\kappa(s) \quad \text{for all } s \in I.$$

(a) **(10 points)** *Prove that*

$$\mathbf{u}(s) = \frac{\lambda \mathbf{t}(s) + \mathbf{b}(s)}{\sqrt{1 + \lambda^2}}$$

*is a constant unit vector.*

**Solution.** The Frenet frame is orthonormal, so

$$|\mathbf{u}|^2 = \frac{\lambda^2 |\mathbf{t}|^2 + 2\lambda \langle \mathbf{t}, \mathbf{b} \rangle + |\mathbf{b}|^2}{1 + \lambda^2} = \frac{\lambda^2 + 1}{1 + \lambda^2} = 1.$$

The Frenet equations (Tapp's Proposition 1.52) give  $\mathbf{t}' = \kappa \mathbf{n}$  and  $\mathbf{b}' = -\tau \mathbf{n}$  (at unit speed). Thus

$$\mathbf{u}' = \frac{\lambda \mathbf{t}' + \mathbf{b}'}{\sqrt{1 + \lambda^2}} = \frac{\lambda \kappa - \tau}{\sqrt{1 + \lambda^2}} \mathbf{n} = 0.$$

Therefore  $\mathbf{u}$  is constant.

**Problem 4, continued.**

The curve  $\gamma$  is unit speed,  $\kappa > 0$ , and  $\tau = \lambda\kappa$  for a fixed  $\lambda \neq 0$ . You may use part (a):

$$\mathbf{u} = \frac{\lambda \mathbf{t}(s) + \mathbf{b}(s)}{\sqrt{1 + \lambda^2}} \quad \text{is a constant unit vector.}$$

(b) **(5 points)** *Prove that the tangent makes a constant angle with  $\mathbf{u}$ .*

**Solution.** By orthonormality,

$$\langle \mathbf{t}(s), \mathbf{u} \rangle = \frac{\lambda \langle \mathbf{t}, \mathbf{t} \rangle + \langle \mathbf{t}, \mathbf{b} \rangle}{\sqrt{1 + \lambda^2}} = \frac{\lambda}{\sqrt{1 + \lambda^2}}.$$

Both vectors  $\mathbf{t}(s)$  and  $\mathbf{u}$  are unit vectors, so  $\langle \mathbf{t}(s), \mathbf{u} \rangle$  is the cosine of the angle made by the two vectors, independent of  $s$ .

(c) **(10 points)** *Prove that  $\gamma$  is injective (one-to-one on  $I$ ). (I.e., if  $\gamma(s_1) = \gamma(s_2)$ , then  $s_1 = s_2$ .)*

[Hint: Show that the function  $h(s) = \langle \gamma(s), \mathbf{u} \rangle$  is one-to-one.]

**Solution.** If we show that  $h(s)$  is one-to-one, then it follows that  $\gamma(s)$  is one-to-one, since if  $\gamma(s_1) = \gamma(s_2)$ , then  $h(s_1) = h(s_2)$ , which implies  $s_1 = s_2$ .

To this end, since  $\mathbf{u}$  is constant,

$$h'(s) = \langle \gamma(s), \mathbf{u} \rangle' = \langle \gamma'(s), \mathbf{u} \rangle = \langle \mathbf{t}(s), \mathbf{u} \rangle = \frac{\lambda}{\sqrt{1 + \lambda^2}}.$$

Since  $h'(s)$  is constant and nonzero,  $h(s)$  is a linear function with nonzero slope, and so is injective.