

Midterm 1

Differential Geometry of Curves and Surfaces

MATH 4230/5230

Fall 2026

Wednesday, September 23, 2026

PRACTICE EXAM

NAME: _____

4 problems; 25 points each; 100 points total.

Problem:	1	2	3	4	Total
Possible:	25	25	25	25	100
Earned:					

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete the exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with this **cover sheet**, and the questions in the correct order.
- You have 50 minutes to complete the exam.

Problem 1**25 points**

Consider the space curve

$$\gamma(t) = \left(t + \frac{t^2}{2} + \frac{t^3}{3}, 1 + t + \frac{t^2}{2} - \frac{t^3}{3}, t^2 \right), \quad -\frac{1}{2} < t < \frac{1}{2}.$$

- (a) **(15 points)** Compute the Frenet frame $\mathbf{t}(0), \mathbf{n}(0), \mathbf{b}(0)$.

Problem 1, continued.

For the same curve

$$\gamma(t) = \left(t + \frac{t^2}{2} + \frac{t^3}{3}, 1 + t + \frac{t^2}{2} - \frac{t^3}{3}, t^2 \right),$$

evaluate the following at $t = 0$.

(b) **(5 points)** *Compute the curvature $\kappa(0)$.*

(c) **(5 points)** *Compute the torsion $\tau(0)$.*

[**Hint:** For any regular space curve with nonzero curvature, one has $\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{|\gamma' \times \gamma''|^2}$.]

Problem 2**25 points**

Let $\gamma : I \rightarrow \mathbb{R}^2$ be a unit-speed plane curve, where $0 \in I$, and suppose $\kappa_s(0) \neq 0$. Let M be the matrix with columns given by the unit tangent vector $\gamma'(0)$ and its rotation by 90° :

$$M = \begin{bmatrix} | & | \\ \gamma'(0) & R_{90}\gamma'(0) \\ | & | \end{bmatrix}.$$

(a) **(10 points)** *Prove that M is orthogonal and $\det M = 1$.*

(b) **(5 points)** Define $f(\mathbf{x}) = M^T(\mathbf{x} - \gamma(0))$ and $\beta(t) = (f \circ \gamma)(t) = M^T(\gamma(t) - \gamma(0))$. By part (a), f is a proper rigid motion, so β is also a unit-speed plane curve. *Prove that $\beta''(0) = (0, \kappa_s(0))$.*

Problem 2, continued.

Recall that γ is unit speed, $\kappa_s(0) \neq 0$, and

$$M = \begin{bmatrix} | & | \\ \gamma'(0) & R_{90}\gamma'(0) \\ | & | \end{bmatrix}, \quad \beta(t) = M^T(\gamma(t) - \gamma(0)).$$

You may use part (b): $\beta''(0) = (0, \kappa_s(0))$.

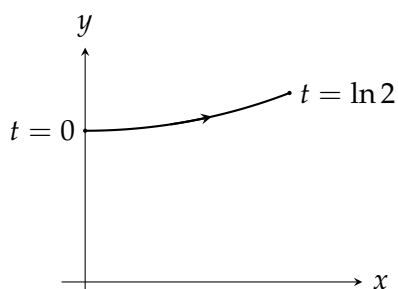
(c) **(10 points)** Find the center of the translated osculating circle of β at $t = 0$, in terms of $\kappa_s(0)$.

[**Hint:** At a point $\mathbf{p}$ of a regular curve with curvature $\kappa > 0$, the translated osculating circle is the circle with center $\mathbf{p} + \frac{1}{\kappa}\mathbf{n}$ and radius $1/\kappa$. Find $\beta(0)$, $\beta'(0)$, and $\beta''(0)$.]

Problem 3**25 points**

Consider the plane curve

$$\gamma(t) = \left(t, \frac{e^t + e^{-t}}{2} \right), \quad 0 \leq t \leq \ln 2.$$



- (a) **(5 points)** Compute the total arc length L of this curve.

Problem 3, continued.

Use the same plane curve

$$\gamma(t) = \left(t, \frac{e^t + e^{-t}}{2} \right), \quad 0 \leq t \leq \ln 2,$$

whose total arc length is denoted by L .

- (b) **(15 points)** Let $\tilde{\gamma} : [0, L] \rightarrow \mathbb{R}^2$ be its orientation-preserving unit-speed reparametrization starting at $\gamma(0)$. Compute $\tilde{\gamma}''(L)$.

[Hint: No inverse function is needed; interpret $\tilde{\gamma}''$ in terms of the curvature, and then compute this from γ .]

- (c) **(5 points)** Let $\mathbf{F}(x, y) = (y + e^x, x)$, and let C be traversed by the given curve from $t = 0$ to $t = \ln 2$. Compute $\int_C \mathbf{F} \cdot d\gamma$.

[Hint: Look for a potential function.]

Problem 4**25 points**

Let $\gamma : I \rightarrow \mathbb{R}^3$ be a unit-speed curve with $\kappa(s) > 0$ and suppose there is a real number λ such that

$$\tau(s) = \lambda \kappa(s) \quad \text{for all } s \in I.$$

(a) **(10 points)** *Prove that*

$$\mathbf{u}(s) = \frac{\lambda \mathbf{t}(s) + \mathbf{b}(s)}{\sqrt{1 + \lambda^2}}$$

is a constant unit vector.

Problem 4, continued.

The curve γ is unit speed, $\kappa > 0$, and $\tau = \lambda\kappa$ for a fixed $\lambda \neq 0$. You may use part (a):

$$\mathbf{u} = \frac{\lambda \mathbf{t}(s) + \mathbf{b}(s)}{\sqrt{1 + \lambda^2}} \text{ is a constant unit vector.}$$

(b) **(5 points)** *Prove that the tangent makes a constant angle with $\mathbf{u}$.*

(c) **(10 points)** *Prove that γ is injective (one-to-one on I). (I.e., if $\gamma(s_1) = \gamma(s_2)$, then $s_1 = s_2$.)*

[**Hint:** Show that the function $h(s) = \langle \gamma(s), \mathbf{u} \rangle$ is one-to-one.]