

MATH 4230/5230: Differential Geometry of Curves and Surfaces

Quiz 4

Friday, September 18, 2026

Name: _____

Score

____/10

Frenet equations. For a unit-speed space curve with curvature $\kappa > 0$, torsion τ , and Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, the Frenet equations are

$$\begin{aligned}\mathbf{t}' &= \kappa \mathbf{n}, \\ \mathbf{n}' &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \mathbf{b}' &= -\tau \mathbf{n}.\end{aligned}$$

Problem (adapted from Tapp, Exercise 1.65). Let $\gamma : I \rightarrow \mathbb{R}^3$ be a unit-speed space curve. Prove that at every $t \in I$ for which $\kappa(t) \neq 0$,

$$\tau(t) = \frac{\langle \gamma'(t) \times \gamma''(t), \gamma'''(t) \rangle}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

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Because γ has unit speed,

$$\gamma' = \mathbf{t}.$$

The first Frenet equation gives

$$\gamma'' = \mathbf{t}' = \kappa \mathbf{n}.$$

Consequently,

$$\gamma' \times \gamma'' = \mathbf{t} \times (\kappa \mathbf{n}) = \kappa(\mathbf{t} \times \mathbf{n}) = \kappa \mathbf{b},$$

and hence

$$\|\gamma' \times \gamma''\|^2 = \kappa^2.$$

Differentiating $\gamma'' = \kappa \mathbf{n}$ and using the second Frenet equation yields

$$\begin{aligned}\gamma''' &= \kappa' \mathbf{n} + \kappa \mathbf{n}' \\ &= \kappa' \mathbf{n} + \kappa(-\kappa \mathbf{t} + \tau \mathbf{b}) \\ &= -\kappa^2 \mathbf{t} + \kappa' \mathbf{n} + \kappa \tau \mathbf{b}.\end{aligned}$$

Since $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is an orthonormal frame,

$$\begin{aligned}\langle \gamma' \times \gamma'', \gamma''' \rangle &= \langle \kappa \mathbf{b}, -\kappa^2 \mathbf{t} + \kappa' \mathbf{n} + \kappa \tau \mathbf{b} \rangle \\ &= \kappa^2 \tau.\end{aligned}$$

Because $\kappa \neq 0$, division by $\|\gamma' \times \gamma''\|^2 = \kappa^2$ gives

$$\boxed{\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{\|\gamma' \times \gamma''\|^2}}.$$