

MATH 4230/5230: Differential Geometry of Curves and Surfaces

Quiz 3

Friday, September 11, 2026

Name: _____

Score ____/10

Definition (signed curvature). Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular plane curve. Write $\mathbf{v}(t) = \gamma'(t)$, $\mathbf{a}(t) = \gamma''(t)$, and $R_{90}(u_1, u_2) = (-u_2, u_1)$. The signed curvature of γ is

$$\kappa_s(t) = \frac{\langle \mathbf{a}(t), R_{90}(\mathbf{v}(t)) \rangle}{\|\mathbf{v}(t)\|^3}.$$

Problem (Tapp, Exercise 1.48). Suppose that $\gamma(t) = (x(t), y(t))$ is a regular plane curve. Prove that

$\kappa_s(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}.$
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Hint. Compute $R_{90}(\mathbf{v}(t))$ and expand the inner product in the definition.

MATH 4230/5230 — Quiz 3 Solutions

For

$$\gamma(t) = (x(t), y(t)),$$

the velocity and acceleration are

$$\mathbf{v}(t) = \gamma'(t) = (x'(t), y'(t)), \quad \mathbf{a}(t) = \gamma''(t) = (x''(t), y''(t)).$$

Rotating the velocity vector counterclockwise by 90° gives

$$R_{90}(\mathbf{v}(t)) = (-y'(t), x'(t)).$$

Therefore,

$$\begin{aligned} \langle \mathbf{a}(t), R_{90}(\mathbf{v}(t)) \rangle &= \langle (x''(t), y''(t)), (-y'(t), x'(t)) \rangle \\ &= -x''(t)y'(t) + y''(t)x'(t) \\ &= x'(t)y''(t) - x''(t)y'(t). \end{aligned}$$

Also,

$$\|\mathbf{v}(t)\|^3 = \left(\sqrt{x'(t)^2 + y'(t)^2} \right)^3 = (x'(t)^2 + y'(t)^2)^{3/2}.$$

Substitution into the definition of signed curvature yields

$$\boxed{\kappa_s(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}}.$$