Reduced row echelon form of a matrix

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August 17, 2025

Introduction

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Putting a matrix in Reduced Row Echelon Form will be the main computational tool we will use in this class.

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$$\begin{bmatrix} \mathbf{1} & 3 & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure: A matrix in reduced row echelon form

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We say that B is obtained from A by elementary row operations if there is a finite sequence of matrices $A = A_0, A_1, \ldots, A_n = B$, with A_{i+1} obtained from A_i, $i = 1, \ldots, n-1$, by an elementary row operation.

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Example (Elementary Row Operations)

$$\begin{bmatrix} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{bmatrix} \quad A = A_0$$

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$$R_1' = \frac{1}{3}R_1 \quad \begin{bmatrix} 1 & 3 & 9 & -1 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{bmatrix} \quad A_1$$

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$$\begin{array}{rcl} & \left[\begin{array}{rrrr} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{array}\right] & A = A_0 \\ R_1' = \frac{1}{3} R_1 & \left[\begin{array}{rrrr} 1 & 3 & 9 & -1 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{array}\right] & A_1 \\ R_2' = R_3 + R_2 & \left[\begin{array}{rrrr} 1 & 3 & 9 & -1 \\ -1 & -3 & -9 & 1 \\ 2 & 8 & 26 & -4 \end{array}\right] & A_2 = B \end{array}$$

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Given a matrix A, there is a unique matrix $RREF(A)$ that is in Reduced Row Echelon Form that can be obtained from A by elementary row operations.

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Exercise.
[Hint: Show that any matrix is row equivalent to a RREF matrix, and then show that if two RREF matrices are row equivalent, then they are equal.] □

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$$\begin{array}{r} \\ R_2' = 3R_1 + R_2 \\ R_3' = -R_1 + R_3 \end{array} \begin{bmatrix} 1 & 3 & 9 & -1 \\ 0 & -2 & -8 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

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$$\begin{array}{l} R_2' = R_3 \\ R_3' = R_2 \end{array} \mapsto \quad R_3'' = 2R_2' + R_3' \begin{bmatrix} 1 & 3 & 9 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$
\begin{array}{rl}
 & \begin{bmatrix} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{bmatrix} \\
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\begin{array}{r} \\ R_2' = R_3 \quad \mapsto \\ R_3' = R_2' \qquad R_3'' = 2R_2 + R_3 \end{array} & \begin{bmatrix} 1 & 3 & 9 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\begin{array}{r} R_1' = R_1 - 3R_2 \\ \\ \\ \end{array} & \begin{bmatrix} \mathbf{1} & 0 & -3 & 2 \\ 0 & \mathbf{1} & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$