

Exercise 4.8.16

Linear Algebra MATH 2130

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ABSTRACT. This is Exercise 4.8.16 from Lay [LLM21, §4.8]:

Exercise 4.8.16. Find a basis for the solution space of the difference equation

$$16y_{k+2} + 8y_{k+1} - 3y_k = 0.$$

Prove that the solutions you find span the solution set.

Solution. To solve this difference equation, we first find roots of the auxiliary equation

$$16z^2 + 8z - 3 = 0.$$

For this, we see that we can factor the quadratic as

$$16z^2 + 8z - 3 = (4z + 3)(4z - 1) = 0$$

so that the roots of $16z^2 + 8z - 3$ are $\alpha_1 = -\frac{3}{4}$ and $\alpha_2 = \frac{1}{4}$. Since this is a degree 2 difference equation, and there are two distinct solutions α_1, α_2 of the auxiliary equation, this means that a basis for the solution space to the difference equation is given by $\{y_k\} = \{\alpha_1^k\}$ and $\{y_k\} = \{\alpha_2^k\}$, or, concretely, the sequences

$$\{y_k\} = \left\{ \left(-\frac{3}{4} \right)^k \right\} \quad \text{and} \quad \{y_k\} = \left\{ \left(\frac{1}{4} \right)^k \right\}$$

form a basis for the solution space to the difference equation $16y_{k+2} + 8y_{k+1} - 3y_k = 0$. Such a statement is not given precisely in the book, and so let us explain here.

First, that both of the sequences above are solutions to difference equation, was stated in class, and is stated in [LLM21, Example 3, p.265]. However, we can also check this easily by hand. If α

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is one of the two solutions α_1 or α_2 to the auxiliary equation, then we have

$$16y_{k+2} + 8y_{k+1} - 3y_k = 16\alpha^{k+2} + 8\alpha^{k+1} - 3\alpha^k = \alpha^k(16\alpha^2 + 8\alpha - 3) = \alpha^k \cdot 0 = 0.$$

If you prefer to check this numerically in each case, say for the case $\{y_k\} = \left\{\left(-\frac{3}{4}\right)^k\right\}$, you can check that

$$\begin{aligned} 16\left(-\frac{3}{4}\right)^{k+2} + 8\left(-\frac{3}{4}\right)^{k+1} - 3\left(-\frac{3}{4}\right)^k &= \left(-\frac{3}{4}\right)^k \left(16\left(-\frac{3}{4}\right)^2 + 8\left(-\frac{3}{4}\right) - 3\right) \\ &= \left(-\frac{3}{4}\right)^k (9 - 6 - 3) = \left(-\frac{3}{4}\right)^k \cdot 0 = 0. \end{aligned}$$

A similar argument shows that $\{y_k\} = \left\{\left(\frac{1}{4}\right)^k\right\}$ is a solution to the difference equation.

In class we asserted that the sequences $\{y_k\} = \{\alpha_1^k\}$ and $\{y_k\} = \{\alpha_2^k\}$ were linearly independent. Since we did not prove this in general, I will provide here a way to check this in this particular example (and in any given example). (This technique is via what is called the Casoratian of the sequences, discussed in the textbook on [LLM21, p.263].) We have the two sequences:

$$\begin{array}{cccc} \dots & 1 & -\frac{3}{4} & \left(-\frac{3}{4}\right)^2 & \dots \\ \dots & 1 & \frac{1}{4} & \left(\frac{1}{4}\right)^2 & \dots \end{array}$$

To show that the two sequences are linearly independent, it suffices to show that the matrix

$$\begin{bmatrix} 1 & -\frac{3}{4} \\ 1 & \frac{1}{4} \end{bmatrix}$$

has linearly independent rows, which is to say, that the reduced row echelon form of the matrix is the identity matrix. It suffices to show that the determinant is non-zero, which is clear (the determinant is 1). So we have proven that the sequences $\{y_k\} = \{\alpha_1^k\}$ and $\{y_k\} = \{\alpha_2^k\}$ are linearly independent.

Finally, to check that the solutions $\{y_k\} = \{\alpha_1^k\}$ and $\{y_k\} = \{\alpha_2^k\}$ span the space of solutions, we use that the space of solutions of a degree 2 difference equation has dimension 2. We stated this in class, and this is also [LLM21, Theorem 20, p.266]. In order to use [LLM21, Theorem 20, p.266], technically, you should observe that the solution space to the difference equation $16y_{k+2} + 8y_{k+1} - 3y_k = 0$ is the same as the solution space to the difference equation obtained by dividing by 16, namely, $y_{k+2} + \frac{8}{16}y_{k+1} - \frac{3}{16}y_k = 0$.

Consequently, since the solution space to the difference equation $16y_{k+2} + 8y_{k+1} - 3y_k = 0$ has dimension 2, and $\{y_k\} = \{\alpha_1^k\}$ and $\{y_k\} = \{\alpha_2^k\}$ are linearly independent solutions to the difference equation, they form a basis for the solution space. $\square$

REFERENCES

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

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