Exercise 4.6.5

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 4.6.5 from Lay [LLM21, §4.6]:

Exercise 4.6.5. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{b}_2, \mathbf{c}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V, and suppose that

$$\begin{array}{rcrcrcr} \mathbf{a}_1 & = & 4\mathbf{b}_1 & - & \mathbf{b}_2 & + & 0\mathbf{b}_3 \\ \mathbf{a}_2 & = & -\mathbf{b}_1 & + & \mathbf{b}_2 & + & \mathbf{b}_3 \\ \mathbf{a}_3 & = & 0\mathbf{c}_1 & + & \mathbf{c}_2 & - & 2\mathbf{c}_3 \end{array} \tag{0.1}$$

a. Find the change-of-coordinates matrix to go from the coordinates with respect to the basis $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ to the coordinates with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

b. Find the coordinates with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for the vector

$$\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3.$$

Solution. a. The change-of-coordinates matrix to go from the coordinates with respect to the basis $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ to the coordinates with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ can be read off from (0.1) as the matrix

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}^T = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

b. The coordinates for the vector

$$\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$$

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with respect to the basis $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ are given by

$$(3,4,1).$$

Therefore, given part a., the coordinates for $\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are given by

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

In other words, the coordinates for $\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are

$$(8,2,2).$$

□

Remark 0.1. Note that in our solution to b., we are claiming that

$$\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3 = 8\mathbf{b}_1 + 2\mathbf{b}_2 + 2\mathbf{b}_3.$$

We could have checked this directly by substituting in the following way:

$$\begin{aligned} \mathbf{x} &= 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3 \\ &= 3(4\mathbf{b}_1 - \mathbf{b}_2 + 0\mathbf{b}_3) + 4(-\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3) + (0\mathbf{d}_1 + \mathbf{b}_2 - 2\mathbf{b}_3) \\ &= 8\mathbf{b}_1 + 2\mathbf{b}_2 + 2\mathbf{b}_3. \end{aligned}$$

References

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

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