Exercise 2.5.24

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 2.5.24 from Lay [LLM21, §2.5]:

Exercise 2.5.24. (*QR factorization*) Suppose that $A = QR$ where Q and R are $n \times n$ matrices, the matrix R is invertible and upper triangular, and Q has the property that $Q^TQ = I$. Show that for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What computations with Q and R will produce the solution?

Solution. To solve the equation $A\mathbf{x} = \mathbf{b}$, we can replace A with QR, to obtain $QR\mathbf{x} = \mathbf{b}$. Then, multiplying both sides by $R^{-1}Q^T$, we can see that

$$\mathbf{x} = (R^{-1}Q^T)(QR)\mathbf{x} = R^{-1}Q^T\mathbf{b}.$$

Therefore, there is exactly one solution to the equation, namely $\mathbf{x} = R^{-1}Q^T\mathbf{b}$. One option to compute this would be to compute R^{-1}, and then compute $R^{-1}Q^T\mathbf{b}$.

Another computational option would be as follows. We could observe that multiplying both sides of the equation $QR\mathbf{x} = \mathbf{b}$ instead by Q^T, we would have

$$R\mathbf{x} = Q^T(QR)\mathbf{x} = Q^T\mathbf{b},$$

so we could compute $Q^T\mathbf{b}$, and then use our standard row reduction algorithm to solve $R\mathbf{x} = Q^T\mathbf{b}$. Since R is upper triangular, these row operations would be relatively easy. □

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References

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

Email address: casa@math.colorado.edu