**Exercise 1.6.11**

**Linear Algebra**
**MATH 2130**

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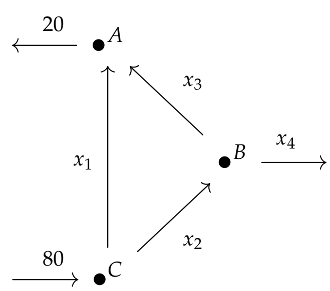
ABSTRACT. This is Exercise 1.6.11 from Lay [LLM21, §1.6]:

**Exercise 1.6.11.** Find the general flow pattern of the network shown in the figure. Assuming the flows are all nonnegative, what is the largest possible value for $x_3$?




*Solution.* We have the following three equations coming from the condition at each node that the total input and output be equal:

$$\begin{array}{rcrcrcrcrcl} x_1 & & & + & x_3 & & & - & 20 & = & 0 \\ & & x_2 & - & x_3 & - & x_4 & & & = & 0 \\ -x_1 & - & x_2 & & & & & + & 80 & = & 0 \end{array}$$

We can rearrange this as:

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & & & = & 20 \\ & & x_2 & - & x_3 & - & x_4 & = & 0 \\ -x_1 & - & x_2 & & & & & = & -80 \end{array}$$

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We then obtain the following augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & -80 \end{array}\right]$$

We now use row operations to put the left hand side of the matrix in RREF:

First we add the first row to the third row:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -60 \end{array}\right]$$

Then we add the second row to the third row:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -60 \end{array}\right]$$

Then we multiply the last row by $-1$, and we also add that row to the second row:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 60 \end{array}\right]$$

The left hand side is in RREF. We then modify the matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} & \color{blue}{0} & \color{blue}{0} \\ 0 & 0 & 0 & 1 & 60 \end{array}\right]$$

The solutions to the system of equations are therefore given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \color{red}{-1} \\ 0 \end{bmatrix} t + \begin{bmatrix} 20 \\ 60 \\ \color{blue}{0} \\ 60 \end{bmatrix}, \quad t \in \mathbb{R}.$$

For the solutions to the problem, we only want those solutions with $x_1, x_2, x_3, x_4 \geq 0$. These are the conditions

$$x_1 = t + 20 \geq 0$$
$$x_2 = -t + 60 \geq 0$$
$$x_3 = -t \geq 0$$
$$x_4 = 60 \geq 0.$$

This translates to

$$t \geq -20$$
$$t \leq 60$$
$$t \leq 0.$$

So we have

$$-20 \leq t \leq 0.$$

Therefore, all possible flows are determined by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 20 \\ 60 \\ 0 \\ 60 \end{bmatrix}, \quad -20 \leq t \leq 0.$$

We can also replace $t$ with $-x_3$, to obtain:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 20 \\ 60 \\ 0 \\ 60 \end{bmatrix}, \quad 0 \leq x_3 \leq 20.$$

The largest possible value for $x_3$ is therefore 20. □

## References

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

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