Exercise 1.5.20

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 1.5.20 from Lay [LLM21, §1.5]:

Exercise 1.5.20. Describe the solutions of the following (inhomogeneous) system in parametric vector form:

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 5x_3 & = & 4 \\ x_1 & + & 4x_2 & - & 8x_3 & = & 7 \\ -3x_1 & - & 7x_2 & + & 9x_3 & = & -6 \end{array}$$

and provide a geometric comparison with the solution set to the (homogeneous) system:

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 5x_3 & = & 0 \\ x_1 & + & 4x_2 & - & 8x_3 & = & 0 \\ -3x_1 & - & 7x_2 & + & 9x_3 & = & 0. \end{array}$$

Solution. The augmented matrix for the first (inhomogeneous) system is

$$\left[\begin{array}{rrr|r} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right]$$

We use row operations to put the left side of the matrix in RREF. First we subtract the first row from the second, and then add 3 times the first row to the third row:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right]$$

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Next we divide the last row by 2:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 1 & -3 & 3 \end{array}\right]$$

Next we subtract the first row from the third:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Next we add -3 times the second row to the first row:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The left hand side is in RREF. We then modify the matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{-1} & \color{blue}{0} \end{array}\right]$$

The solutions to the first (inhomogeneous) system are then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ \color{red}{-1} \end{bmatrix} t + \begin{bmatrix} -5 \\ 3 \\ \color{blue}{0} \end{bmatrix}, \quad t \in \mathbb{R}.$$

The augmented matrix for the second (homogeneous) system is:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{array}\right]$$

A similar computation then shows that the solutions to the second (homogeneous) system are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} t, \quad t \in \mathbb{R}.$$

In other words, the solutions to the first (inhomogeneous) system are the solutions to the second (homogeneous) system, shifted by the vector (particular solution) $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$. □

Remark 0.1. In the above, we are free to replace t with $-x_3$, and the solutions to the first (inhomogeneous) system would then be given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R},$$

and the solutions to the second (homogeneous) system would then be given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} x_3, \quad x_3 \in \mathbb{R}.$$

It is always a good idea to check our solutions. We said that the solutions to the first (inhomogeneous) system were given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} t + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

First we should check the solution given by $t = 0$, namely

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

Plugging in, we see we are confirming that

$$\begin{array}{rcrcrcr} (-5) & + & 3(3) & - & 5(0) & \overset{?}{=} & 4 \\ (-5) & + & 4(3) & - & 8(0) & \overset{?}{=} & 7 \\ -3(-5) & - & 7(3) & + & 9(0) & \overset{?}{=} & -6 \end{array}$$

Since the above hold, we did indeed have a solution (to the inhomogeneous system).

Next we check that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$$

is a solution to the second (homogeneous) system of equations. Plugging in, we see we are confirming:

$$\begin{array}{rcrcrcl} (4) & + & 3(-3) & - & 5(-1) & \stackrel{?}{=} & 0 \\ (4) & + & 4(-3) & - & 8(-1) & \stackrel{?}{=} & 0 \\ -3(4) & - & 7(-3) & + & 9(-1) & \stackrel{?}{=} & 0 \end{array}$$

Since the above hold, we see that we do have a solution (to the homogeneous system).

Together, this implies that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} t + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

are solutions to the first (inhomogeneous) system.

Remark 0.2. Note that this only confirms that these are some of the solutions to the system, not that these are *all* of the solutions. These are in fact all of the solutions, as we proved above. I am just pointing out that our *check* here only confirms that the solutions we proposed were in fact solutions (not that they were *all* of the solutions).

References

[LLM21] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Sixth edition, Pearson, 2021.

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