Midterm I

Intro to Discrete Math

MATH 2001

Fall 2024

Friday September 27, 2024

NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	25	15	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 45 minutes to complete the exam. **We will spend the last 5 minutes of class to upload your exam to Canvas.**

1. • Consider the sets $A = \{3, 9\}$ and $B = \{2, 3, 5\}$.

For this problem **you do <u>not</u> need to justify your answer**.

(a) (5 points) *List the elements of the power set $\mathscr{P}(A)$.*

SOLUTION:

$$\boxed{\emptyset, \{3\}, \{9\}, \{3, 9\}}$$

(b) (5 points) *List the elements of the set $A \times B$.*

SOLUTION:

$$\boxed{(3, 2), (3, 3), (3, 5), (9, 2), (9, 3), (9, 5).}$$

(c) (5 points) *List the elements of the set $A \cup B$.*

SOLUTION:

$$\boxed{2, 3, 5, 9}$$

(d) (5 points) *Is it true that $2 \in (A \cap B)$?*

SOLUTION:

$$\boxed{\text{No}}$$

(e) (5 points) *List the elements of the set $B - A$.*

SOLUTION:

$$\boxed{2, 5}$$

1
25 points

2. (15 points) • Suppose that A and B are finite sets. *What is $|A \times B|$ in terms of $|A|$ and $|B|$?* Explain.

SOLUTION:

Solution.

$$\boxed{|A \times B| = |A||B|}$$

Indeed, recall that the product $A \times B$ is the set of ordered pairs:

$$A \times B := \{(a, b) : a \in A,\ b \in B\}.$$

There are $|A|$ choices for the first entry of an ordered pair (a, b) in the product $A \times B$, and then for each choice of first entry there are $|B|$ choices for the second entry, so that all together, we have $|A||B|$ ordered pairs in the product $A \times B$. □

You can find a similar explanation on p.9 of Hammack, where this is Fact 1.1.

Also, as an example to illustrate the formula, suppose that $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. Then we have

$$A \times B = \{(a_1, b_1),\ (a_1, b_2),\ (a_2, b_1),\ (a_2, b_2),\ (a_3, b_1),\ (a_3, b_2)\},$$

and so we can count that $|A \times B| = 6 = 3 \cdot 2 = |A||B|$, confirming the general formula in this special case. (See also Problem 1(b) for a similar example.)

2
15 points

3. (20 points) • For each $t \in \mathbb{R}$, consider the set

$$A_t = \{(x, y) \in \mathbb{R}^2 : t \leq x \leq t + 1 \text{ and } y = t\}.$$

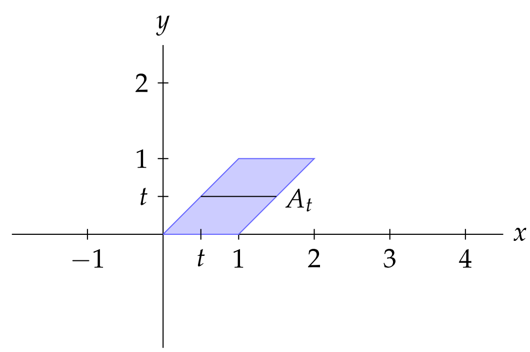
Describe the union

$$\bigcup_{t \in [0,1]} A_t$$

as a geometric object in the plane $\mathbb{R}^2$. A good picture and a brief explanation of your solution is sufficient for this problem.

SOLUTION:

Solution. For each $t \in \mathbb{R}$, the set A_t is the horizontal line segment joining the points (t, t) and $(t + 1, t)$.



Therefore, the union of the A_t over all t in the interval $[0, 1]$ is the region bounded by the parallelogram with vertices $(0, 0)$, $(1, 0)$, $(2, 1)$, and $(1, 1)$. □

3
20 points

4. • Let $a \in \mathbb{R}$ and let f be a function from $\mathbb{R}$ to $\mathbb{R}$ (i.e., a function just like you are used to in all of your math classes, such as $f(x) = x^2$, $f(x) = \sin(x)$, etc.).

(a) (10 points) *Write the following sentence using only logical and mathematical notation* (e.g., the symbols $\wedge, \forall, \exists, \implies, \in, \mathbb{R}$, etc.). You may find it convenient to use the set $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$.

For all real numbers M, there exists a positive real number δ such that for all real numbers x, if $0 < |x - a| < \delta$, then $f(x)$ is greater than M.

SOLUTION:

$$\boxed{\forall M \in \mathbb{R},\ \exists \delta \in \mathbb{R}^+ \ s.t.\ \forall x \in \mathbb{R},\ (0 < |x - a| < \delta) \implies (f(x) > M).}$$

(b) (5 points) *Negate your answer to the previous part and simplify it as much as you can using standard logical equivalences.*

SOLUTION:

$$\sim (\forall M \in \mathbb{R},\ \exists \delta \in \mathbb{R}^+ \ s.t.\ \forall x \in \mathbb{R},\ (0 < |x - a| < \delta) \implies (f(x) > M))$$

$$\exists M \in \mathbb{R} \ s.t.\ \sim (\exists \delta \in \mathbb{R}^+ \ s.t.\ \forall x \in \mathbb{R},\ (0 < |x - a| < \delta) \implies (f(x) > M))$$

$$\exists M \in \mathbb{R} \ s.t.\ \forall \delta \in \mathbb{R}^+,\ \sim (\forall x \in \mathbb{R},\ (0 < |x - a| < \delta) \implies (f(x) > M))$$

$$\exists M \in \mathbb{R} \ s.t.\ \forall \delta \in \mathbb{R}^+,\ \exists x \in \mathbb{R} \ s.t.\ \sim ((0 < |x - a| < \delta) \implies (f(x) > M))$$

$$\boxed{\exists M \in \mathbb{R} \ s.t.\ \forall \delta \in \mathbb{R}^+,\ \exists x \in \mathbb{R} \ s.t.\ (0 < |x - a| < \delta) \wedge \sim ((f(x) > M)}$$

or

$$\boxed{\exists M \in \mathbb{R} \ s.t.\ \forall \delta \in \mathbb{R}^+,\ \exists x \in \mathbb{R} \ s.t.\ (0 < |x - a| < \delta) \wedge (f(x) \leq M)}$$

(c) (5 points) *Rewrite your answer to the previous part in plain english.*

SOLUTION:

There exists a real number M such that for all positive real numbers δ, there exists a real number x such that $0 < |x - a| < \delta$ and $f(x)$ is not greater than M.

or

There exists a real number M such that for all positive real numbers δ, there exists a real number x such that $0 < |x - a| < \delta$ and $f(x)$ is less than or equal to M.

4
20 points

5. • **True or False**. For this problem **you do not need to justify your answer**.

(a) (4 points) **True or False** (circle one). If A and B are finite sets and $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

SOLUTION: TRUE. More generally, for finite sets A and B, we have $|A \cup B| = |A| + |B| - |A \cap B|$.

(b) (4 points) **True or False** (circle one). If A and B are subsets of a set X, then, regarding complements of subsets of X, we have $(A \cap B)^C = A^C \cup B^C$.

SOLUTION: TRUE. We explained this in class.

(c) (4 points) **True or False** (circle one). The negation of the statement $\forall x \in X, \exists y \in Y$ *s.t.* $p(x,y)$ is logically equivalent to the statement $\exists x \in X$ *s.t.* $\exists y \in Y$ *s.t.* $\sim p(x,y)$.

SOLUTION: FALSE. The negation is logically equivalent to $\exists x \in X$ *s.t.* $\forall y \in Y, \sim p(x,y)$. To see that $\exists x \in X$ *s.t.* $\exists y \in Y$ *s.t.* $\sim p(x,y)$ is NOT the correct negation, consider the example where $X = \mathbb{Z} - \{0\}$, $Y = \mathbb{Q}$, and for $x \in X$ and $y \in Y$, we take $p(x,y)$ to be the statement $xy = 1$ (then both the statement $\forall x \in X, \exists y \in Y$ *s.t.* $p(x,y)$ and the claimed negation, $\exists x \in X$ *s.t.* $\exists y \in Y$ *s.t.* $\sim p(x,y)$, would be true, which is not possible).

(d) (4 points) **True or False** (circle one). The truth table for the statement $p \wedge q$ is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	T
F	F	F

SOLUTION: FALSE. The third row is not correct: if p is false and q is true, then $p \wedge q$ if false.

(e) (4 points) **True or False** (circle one). $(\mathbb{R} - \mathbb{Z}) \times \mathbb{N} = (\mathbb{R} \times \mathbb{N}) - (\mathbb{Z} \times \mathbb{N})$.

SOLUTION: TRUE. In fact, more generally if A, B, and C are sets, then we have the equality $(A - B) \times C = (A \times C) - (B \times C)$.

5
20 points