Midterm II

Intro to Discrete Math

MATH 2001

Fall 2024

Monday November 4, 2024

NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with this **cover sheet**, and the questions in the correct order.
- You have 45 minutes to complete the exam. **We will spend the last 5 minutes of class to upload your exam to Canvas.**

1. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do <u>not</u> need to justify your answer**.

(a) (4 points) **TRUE** or **FALSE** (circle one). For any natural number n, we have $5^n = \sum_{k=0}^{n} \binom{n}{k} 4^k$.

SOLUTION: TRUE. By the Binomial Theorem we have $5^n = (1+4)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 4^k$.

(b) (4 points) **TRUE** or **FALSE** (circle one). If A, B, and C are finite sets that are pairwise disjoint (i.e., $A \cup B \cup C = A \sqcup B \sqcup C$), then $|A \cup B \cup C| = |A| + |B| + |C|$.

SOLUTION: TRUE. This is called the Addition Principle in the book (see Fact 3.2, p.74).

(c) (4 points) **TRUE** or **FALSE** (circle one). If n is a natural number, then $n!$ is the number of lists of length n that can be made from n symbols, without repetition.

SOLUTION: TRUE. (see Definition 3.1, p.78).

(d) (4 points) **TRUE** or **FALSE** (circle one). Suppose a multiset A has n elements, with multiplicities $p_1, p_2, \ldots, p_k$. Then the total number of permutations of A is $\binom{n!}{(p_1 p_2 \cdots p_k)!}$.

SOLUTION: FALSE. Consider the multiset $[1,1,2,2]$. There are 6 permutations, $[1,1,2,2]$, $[1,2,1,2]$, $[1,2,2,1]$, and $[2,1,1,2]$, $[2,1,2,1]$, and $[2,2,1,1]$, but $\binom{n!}{(p_1 p_2 \cdots p_k)!} = \binom{4!}{(2 \cdot 2)!} = 1$. The correct formula is $\frac{n!}{p_1! p_2! \cdots p_k!}$. This is Fact 3.8 on p.102 (there are $n!$ permutations of the elements, but we cannot distinguish the $p_1!$ permutations of the first elements, etc.).

(e) (4 points) **TRUE** or **FALSE** (circle one). Suppose n objects are placed into k boxes. Then at least one box contains $\lfloor \frac{n}{k} \rfloor$ or more objects, and at least one box contains $\lceil \frac{n}{k} \rceil$ or fewer objects.

SOLUTION: TRUE. However, one can do better; the Division Principle (Fact 3.9, p.104) states that at least one box contains $\lceil \frac{n}{k} \rceil$ or more objects, and at least one box contains $\lfloor \frac{n}{k} \rfloor$ or fewer objects.

1
20 points

2. (20 points) • For a five card hand from a standard deck of cards, a *full house* is a hand that has three cards of one number and two cards of another number. For instance $Q♣$, $Q\diamondsuit$, $Q\heartsuit$, $3\heartsuit$, $3♠$ is a full house. (Here, as in class, we are numbering the cards, with ace being 1, jack being 11, queen being 12, and king being 13.)

TRUE or **FALSE**:

$$\textit{There are } (13 \cdot 12) \cdot \binom{4}{3} \cdot \binom{4}{2} \textit{ full houses in a standard deck of cards.}$$

If true, give a short *proof* of the statement. If false, give a short *proof* that the statement is false. Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

[*Hint*: How many five card hands have three queens and two threes?]

SOLUTION:

Solution. This statement is TRUE.

Given a hand that is a full house, there are three cards of one number and two cards of another. So, to count the number of full houses, we can start by choosing what the first number is (the one appearing 3 times) and what the second number is (the number appearing 2 times). For instance, the hand $Q♣$, $Q\diamondsuit$, $Q\heartsuit$, $3\heartsuit$, $3♠$ would correspond to choosing a queen for the first number and a three for the second number. The order matters, so there are

$$13 \cdot 12$$

ways to choose the two numbers.

Once we have fixed the first number, then, since there are four cards with that number in the deck (one for each suit), there are $\binom{4}{3}$ ways to choose three cards of that number, and once we have fixed the second number, there are $\binom{4}{2}$ ways to choose two cards of that number. For instance, there are $\binom{4}{3} \cdot \binom{4}{2}$ five card hands with three queens and two threes.

All together, this gives us

$$(13 \cdot 12) \cdot \binom{4}{3} \cdot \binom{4}{2}$$

full houses in a standard deck of cards. □

2
20 points

3. (20 points) • Let n, k, and ℓ be natural numbers with $2 \le k \le \ell \le n$. *Use a combinatorial proof to show*

$$\binom{n}{2}\binom{n-2}{k-2}\binom{n-k}{\ell-k} = \binom{n}{\ell}\binom{\ell}{k}\binom{k}{2}.$$

[*Hint*: You may want to consider sets $A \subseteq B \subseteq C$.]

SOLUTION:

Solution. Let $S = \{1, \ldots, n\}$ and consider the set

$$T = \{(A, B, C) \in \mathscr{P}(S) \times \mathscr{P}(S) \times \mathscr{P}(S) : |A| = 2,\ |B| = k,\ |C| = \ell,\ A \subseteq B \subseteq C\}.$$

In other words, T is the set of ordered triples of subsets of $S = \{1, \ldots, n\}$ such that the first subset has 2 elements, the second subset has k elements, the third subset has ℓ elements, the first subset is contained in the second subset, and the second subset is contained in the third.

We will count the number of elements of T in two ways.

First, we can fix the set $A \subseteq \{1, \ldots, n\}$ with $|A| = 2$. There are $\binom{n}{2}$ ways to do this. Then we can choose B with $|B| = k$ and $B \supseteq A$. Since we already have the 2 elements of A inside of B, we must choose $k-2$ elements from the remaining $n-2$ elements of $\{1, \ldots, n\}$ to determine B. There are $\binom{n-2}{k-2}$ ways to do this. Finally, we can choose C with $|C| = \ell$ and $C \supseteq B$. Since we already have the k elements of B inside of C, we must choose $\ell - k$ elements from the remaining $n-k$ elements of $\{1, \ldots, n\}$ to determine C. There are $\binom{n-k}{\ell-k}$ ways to do this. Therefore

$$|T| = \binom{n}{2}\binom{n-2}{k-2}\binom{n-k}{\ell-k}.$$

On the other hand, we can first fix the set $C \subseteq \{1, \ldots, n\}$ with $|C| = \ell$. There are $\binom{n}{\ell}$ ways to do this. Then we can determine $B \subseteq C$ with $|B| = k$ by fixing k elements of C; there are $\binom{\ell}{k}$ ways to do this. Then we can determine $A \subseteq B$ with $|A| = 2$ by fixing 2 elements of B; there are $\binom{k}{2}$ ways to do this. Therefore

$$|T| = \binom{n}{\ell}\binom{\ell}{k}\binom{k}{2}.$$

In conclusion, we have

$$\binom{n}{2}\binom{n-2}{k-2}\binom{n-k}{\ell-k} = \binom{n}{\ell}\binom{\ell}{k}\binom{k}{2}.$$

□

3
20 points

4. (20 points) • **TRUE** or **FALSE**:

$$\textit{If } n \in \mathbb{N}, \textit{ then } \binom{2n}{n} \textit{ is even.}$$

If true, give a *direct proof* of the statement. If false, provide a *counter example*, and prove that it is a counter example. Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

SOLUTION:

Solution. This statement is TRUE. Indeed, we have

$$\begin{aligned}\binom{2n}{n} &= \frac{2n!}{n!(2n-n)!} = \frac{2n!}{n!n!} = \frac{(2n)(2n-1)(2n-2)\cdots(n+2)(n+1)}{n(n-1)(n-2)\cdots 2\cdot 1} \\ &= \frac{2n}{n}\cdot\frac{(2n-1)(2n-2)\cdots(n+2)(n+1)}{(n-1)(n-2)\cdots 2\cdot 1} \\ &= 2\cdot\binom{2n-1}{n},\end{aligned}$$

which is even, since $\binom{2n-1}{n}$ is an integer. □

4
20 points

5. (20 points) • In class we showed that the equation $x^2 + y^2 = 3$ has no rational solutions. Use this fact to give a *proof by contradiction* of the statement:

If k is an odd positive integer, then the equation $x^2 + y^2 = 3^k$ has no rational solutions.

SOLUTION:

Solution. Let k be an odd positive integer, and suppose for the sake of contradiction that there exists a rational solution $(x_0, y_0) \in \mathbb{Q}^2$ to the equation $x^2 + y^2 = 3^k$. In other words, we assume that there exist rational numbers x_0 and y_0 such that

$$x_0^2 + y_0^2 = 3^k.$$

Since k is odd, we have that $k = 2n + 1$ for some non-negative integer n. Substituting, we have

$$x_0^2 + y_0^2 = 3^{2n+1}.$$

If we divide both sides of the equation above by 3^{2n}, we obtain the equation

$$\left(\frac{x_0}{3^n}\right)^2 + \left(\frac{y_0}{3^n}\right)^2 = 3.$$

Since $x_0/3^n$ and $y_0/3^n$ are rational numbers, the equation above would say that $(x_0/3^n, y_0/3^n)$ gives a rational solution to the equation $x^2 + y^2 = 3$, which we know is impossible. Consequently, since we have arrived at a contradiction, our assumption that there exists a rational solution $(x_0, y_0) \in \mathbb{Q}^2$ to the equation $x^2 + y^2 = 3^k$ with k an odd integer was false.

Therefore, if k is an odd positive integer, then the equation $x^2 + y^2 = 3^k$ has no rational solutions. □

5
20 points