Midterm II

Intro to Discrete Math

MATH 2001

Fall 2024

Monday November 4, 2024

NAME: ______________________________

PRACTICE EXAM

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with this **cover sheet**, and the questions in the correct order.
- You have 45 minutes to complete the exam. **We will spend the last 5 minutes of class to upload your exam to Canvas.**

1. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do <u>not</u> need to justify your answer**.

(a) (4 points) **TRUE** or **FALSE** (circle one). For any natural number n, we have $5^n = \sum_{k=0}^{n} \binom{n}{k} 4^k$.

(b) (4 points) **TRUE** or **FALSE** (circle one). If A, B, and C are finite sets that are pairwise disjoint (i.e., $A \cup B \cup C = A \sqcup B \sqcup C$), then $|A \cup B \cup C| = |A| + |B| + |C|$.

(c) (4 points) **TRUE** or **FALSE** (circle one). If n is a natural number, then $n!$ is the number of lists of length n that can be made from n symbols, without repetition.

(d) (4 points) **TRUE** or **FALSE** (circle one). Suppose a multiset A has n elements, with multiplicities $p_1, p_2, \ldots, p_k$. Then the total number of permutations of A is $\left(\frac{n!}{(p_1 p_2 \cdots p_k)!}\right)$.

(e) (4 points) **TRUE** or **FALSE** (circle one). Suppose n objects are placed into k boxes. Then at least one box contains $\lfloor \frac{n}{k} \rfloor$ or more objects, and at least one box contains $\lceil \frac{n}{k} \rceil$ or fewer objects.

1
20 points

2. (20 points) • For a five card hand from a standard deck of cards, a *full house* is a hand that has three cards of one number and two cards of another number. For instance $K\clubsuit$, $K\diamondsuit$, $K\heartsuit$, $3\heartsuit$, $3\spadesuit$ is a full house. (Here, as in class, we are numbering the cards, with ace being 1, jack being 11, queen being 12, and king being 13.)

TRUE or **FALSE**:

$$\textit{There are } (13 \cdot 12) \cdot \binom{4}{3} \cdot \binom{4}{2} \textit{ full houses in a standard deck of cards.}$$

If true, give a short *proof* of the statement. If false, give a short *proof* that the statement is false. Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

[*Hint*: How many five card hands have three kings and two threes?]

2
20 points

3. (20 points) • Let n, k, and ℓ be natural numbers with $2 \le k \le \ell \le n$. *Use a combinatorial proof to show*

$$\binom{n}{2}\binom{n-2}{k-2}\binom{n-k}{\ell-k} = \binom{n}{\ell}\binom{\ell}{k}\binom{k}{2}.$$

[*Hint*: You may want to consider sets $A \subseteq B \subseteq C$.]

3
20 points

4. (20 points) • **TRUE** or **FALSE**:

$$\textit{If } n \in \mathbb{N}, \textit{ then } \binom{2n}{n} \textit{ is even.}$$

If true, give a *direct proof* of the statement. If false, provide a *counter example*, and prove that it is a counter example. Your solution must start with the sentence, "*This statement is TRUE*," or the sentence, "*This statement is FALSE*."

4
20 points

5. (20 points) • In class we showed that the equation $x^2 + y^2 = 3$ has no rational solutions. Use this fact to give a *proof by contradiction* of the statement:

If k is an odd positive integer, then the equation $x^2 + y^2 = 3^k$ has no rational solutions.

5
20 points