Exercise 4.7

Introduction to Discrete Mathematics
MATH 2001

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 4.7 from Hammack [Ham13, Ch. 4]:

Exercise 4.7. Use the method of direct proof to prove the following statement: *Suppose a is an integer. If $7 \mid 4a$, then $7 \mid a$.*

Solution. Suppose that a is an integer, and that 7 divides $4a$. Using unique factorization of integers into products of primes, it follows that since 7 is a prime number, and 7 does not divide 4, it must be that 7 divides a.

More precisely, the fact that 7 divides $4a$ means, by definition, that there exists an integer n such that $4a = 7n$. Every integer integer has a unique factorization into primes; in other words, we may write $a = (-1)^{r_{-1}} 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7} \cdots$, where $r_{-1} \in \{0,1\}$ and $r_2, r_3, r_5, r_7, \ldots$ are non-negative integers, with all but finitely many equal to zero, and this expression is unique. Similarly, we can write $n = (-1)^{s_{-1}} 2^{s_2} 3^{s_3} 5^{r_5} 7^{r_7} \cdots$, where $s_{-1} \in \{0,1\}$ and $s_2, s_3, s_5, s_7, \ldots$ are non-negative integers, with all but finitely many equal to zero, and this expression is unique.

Using these prime factorizations, the equation $4a = 7n$ can be written as

$$2^2 \cdot (-1)^{r_{-1}} 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7} \cdots = 7 \cdot (-1)^{s_{-1}} 2^{s_2} 3^{s_3} 5^{s_5} 7^{s_7} \cdots$$

Since prime factorizations are unique, this means that $r_7 = s_7 + 1 \geq 1$, which implies $r_7 - 1 \geq 0$. Therefore, writing $a = (-1)^{r_{-1}} 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7} \cdots = 7 \cdot (-1)^{r_{-1}} 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7 - 1} \cdots$, we have that 7 divides a, since the fact that $r_7 - 1 \geq 0$ implies that $(-1)^{r_{-1}} 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7 - 1} \cdots$ is an integer. □

Date: February 27, 2022.

References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

Email address: casa@math.colorado.edu