Exercise 3.10.3

Introduction to Discrete Mathematics
MATH 2001

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ABSTRACT. This is Exercise 3.10.3 from Hammack [Ham13, §3.10]:

Exercise 3.10.3. Let n and k be natural numbers with $2 \leq k \leq n$. Use a combinatorial proof to show that $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

Solution. For this, let us consider the set

$$T = \{(A, B) \in \mathscr{P}(\{1, \ldots, n\}) \times \mathscr{P}(\{1, \ldots, n\}) : |A| = 2,\ |B| = k,\ A \subseteq B\}.$$

In other words, T is the set of ordered pairs of subsets of $\{1, \ldots, n\}$ such that the first subset has 2 elements, the second subset has k elements, and the first subset is contained in the second subset. We will count the number of elements of T in two ways.

First, we can fix the set $A \subseteq \{1, \ldots, n\}$ with $|A| = 2$. There are $\binom{n}{2}$ ways to do this. Then we can choose B with $|B| = k$ and $B \supseteq A$. Since we already have the 2 elements of A inside of B, we must choose $k - 2$ elements from the remaining $n - 2$ elements of $\{1, \ldots, n\}$ to determine B. There are $\binom{n-2}{k-2}$ ways to do this. Therefore

$$|T| = \binom{n}{2}\binom{n-2}{k-2}.$$

On the other hand, we can first fix the set $B \subseteq \{1, \ldots, n\}$ with $|B| = k$. There are $\binom{n}{k}$ ways to do this. Then we can determine $A \subseteq B$ with $|A| = 2$ by fixing 2 elements of B; there are $\binom{k}{2}$ ways to do this. Therefore

$$|T| = \binom{n}{k}\binom{k}{2}.$$

In conclusion, we have

$$\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}.$$

□

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References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

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