Exercise 3.10.2

Introduction to Discrete Mathematics
MATH 2001

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ABSTRACT. This is Exercise 3.10.2 from Hammack [Ham13, §3.10]:

Exercise 3.10.2. Let n be a natural number. Use a combinatorial proof to show that

$$1 + 2 + \cdots + n = \binom{n+1}{2}.$$

Solution. The right hand side of the equation, $\binom{n+1}{2}$, is the number of ways of choosing a two element subset from the set $\{1, \ldots, n+1\}$. Let us count these subsets in another way.

For this, let $T \subseteq \mathscr{P}(\{1, \ldots, n+1\})$ be the set of two element subsets. For each $i = 1, \ldots, n$, let $T_i \subseteq T$ be the set of two element subsets $S \subseteq \{1, \ldots, n+1\}$ such that i is the smallest element of S. For instance, we have $T_n = \{\{n, n+1\}\}$ and $T_{n-1} = \{\{n-1, n\}, \{n-1, n+1\}\}$. We clearly have

$$T = \bigsqcup_{i=1}^{n} T_i.$$

Moreover, we also have that $|T_i| = n + 1 - i$, since once we set the smallest element of a two element set to be i, then the second element must be chosen from $i+1, i+2, \ldots, n+1$, and there are $n + 1 - i$ of these. Therefore we have

$$\binom{n+1}{2} = |T| = \sum_{i=1}^{n} |T_i| = \sum_{i=1}^{n} n + 1 - i = n + (n-1) + \cdots + 2 + 1.$$

□

Remark 0.1. Another standard proof of the fact that $\sum_{i=1}^{n} i = \binom{n+1}{2}$ is to consider the following table:

1	2	$\cdots$	n
n	$n-1$	$\cdots$	1

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The sum of each row is $\sum_{i=1}^{n} i$. The sum of each column is $n+1$. Therefore, adding up both rows, we get $2\sum_{i=1}^{n} i$ and adding up the n columns we get $n(n+1)$. Therefore, $2\sum_{i=1}^{n} i = n(n+1)$, and dividing by two gives the result.

References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

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