Exercise 14.3.8

Introduction to Discrete Mathematics
MATH 2001

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ABSTRACT. This is Exercise 14.3.8 from Hammack [Ham13, §14.3]:

Exercise 14.3.8. Prove or disprove: *The set* $\{a_1, a_2, a_3, \cdots : a_i \in \mathbb{Z}\}$ *of infinite sequences of integers is countably infinite.*

Solution. The set $\{a_1, a_2, a_3, \cdots : a_i \in \mathbb{Z}\}$ of infinite sequences of integers is not countable, and so in particular, is not countably infinite. Recall that in class we proved that the subset

$$\{a_1, a_2, a_3, \cdots : a_i \in \{0,1\}\} \subseteq \{a_1, a_2, a_3, \cdots : a_i \in \mathbb{Z}\}$$

consisting of infinite sequences of zeros and ones is not countable. More precisely saw that the set $\{a_1, a_2, a_3, \cdots : a_i \in \{0,1\}\}$ is in bijection with the set of maps $\mathrm{Map}(\mathbb{N}, \{0,1\})$ from $\mathbb{N}$ to $\{0,1\}$, which is in bijection with the power set $\mathscr{P}(\mathbb{N})$, which we proved was not countable by showing that there is no surjective map of sets $\mathbb{N} \to \mathscr{P}(\mathbb{N})$.

This proves that $\{a_1, a_2, a_3, \cdots : a_i \in \mathbb{Z}\}$ is not countable, since any subset of a countable set is countable, so that if $\{a_1, a_2, a_3, \cdots : a_i \in \mathbb{Z}\}$ were countable, then $\{a_1, a_2, a_3, \cdots : a_i \in \{0,1\}\}$ would be countable, giving a contradiction. □

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References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

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