CHAPTER 177

# Review of complex analysis in one variable

This gives a brief review of some of the basic results in complex analysis. In particular, it outlines the background in single variable complex analysis that is discussed in [Huy05, §1.1].

## 1. Complex numbers

We define the complex numbers $\mathbb{C}$ to be the field $(\mathbb{R}^2, +, \cdot)$ where $(\mathbb{R}^2, +)$ is the standard $\mathbb{R}$-vector space of dimension 2, and $\cdot$ is defined by $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. For convenience write $(a, b) = a + ib$. We will denote by $\mathrm{CO}(2, \mathbb{R})$ the group of real two by two conformal matrices:

$$\mathrm{CO}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : (a, b) \in \mathbb{R}^2 - \{0\} \right\}.$$

Set

$$\widehat{\mathrm{CO}}(2, \mathbb{R}) = \mathrm{CO}(2, \mathbb{R}) \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a ring under matrix addition and multiplication.

**Exercise 177.1.1.** *Show that there is an isomorphism of rings*

$$\phi : \mathbb{C} \to \widehat{\mathrm{CO}}(2, \mathbb{R})$$
$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

**Exercise 177.1.2.** *Given a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$, there exists a linear map $\alpha \in M(1, \mathbb{C}) = \mathbb{C}$ making the following diagram commute*

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \| & & \| \\ \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C}. \end{array}$$

*if and only if $A \in \widehat{\mathrm{CO}}(2, \mathbb{R})$. In this case $A = \phi(\alpha)$. In particular, given $a + ib \in \mathbb{C}$, then multiplication of complex numbers by $a + ib$, when viewed as an $\mathbb{R}$-linear map of $\mathbb{R}^2$, is given by $\phi(a + ib)$.*

## 2. Holomorphic maps

**Definition 177.2.3** (Holomorphic map)**.** *Let $U \subseteq \mathbb{C}$ be an open subset. A map*

$$f : U \to \mathbb{C}$$

*is said to be holomorphic if it at each point $p \in U$, the real differential $D_p f : \mathbb{R}^2 \to \mathbb{R}^2$ exists and is complex linear (i.e., $D_p f \in \widehat{\mathrm{CO}}(2, \mathbb{R})$).*

EXAMPLE 177.2.4. A complex analytic function on an open subset of the complex plane is holomorhpic (on that open subset). We will recall below the proof that the converse holds.

EXAMPLE 177.2.5. In particular, the function $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is analytic on $\mathbb{C}$, and is therefore holomorphic.

**Corollary 177.2.6** (Cauchy–Riemann). *Let $U \subseteq \mathbb{C} = \mathbb{R}^2$ be an open subset. A map $f : U \to \mathbb{C}$ that is differentiable at each point $p \in U$ is holomorphic if and only if, writing $f(x,y) = u(x,y) + iv(x,y)$, the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p), \quad \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$$

*hold at each point $p \in U$.*

PROOF. This follows immediately from the definitions. □

REMARK 177.2.7. The Cauchy–Riemann equations imply that if we define $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, then a differentiable function $f(z) = u(x,y) + iv(x,y)$ is holomorphic on an open set $U$ if and only if $\frac{\partial}{\partial \bar{z}} f(z) = 0$ for every $z \in U$.

Recall that if $\Gamma$ is a (positively oriented) smooth contour in the complex plane, parameterized by a smooth map $\gamma : [a,b] \to \mathbb{C}$, then

$$\int_{\Gamma} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

EXAMPLE 177.2.8. The main example is:

$$\int_{\partial B_\epsilon(0)} z^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1, \end{cases}$$

where $B_\epsilon(0)$ is the ball of radius $\epsilon > 0$ around the origin.

The main bound that one uses repeatedly is:

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$$\left|\int_{\Gamma} f(z)dz\right| \leq \int_{\Gamma} |f(z)|ds \leq \sup_{z\in\Gamma} |f(z)||\Gamma| \tag{88}$$

where $|\Gamma|$ is the length of the path $\Gamma$ (e.g., [Rud87, §10.8 Eq. (5), p.202]). [1] Recall that for any continuous function $h : \mathbb{R}^2 \to \mathbb{R}$, we have $\int_{\Gamma} h ds = \int_a^b h(\gamma(t))|\gamma'(t)|dt$, and the length of $\Gamma$ is $\int_{\Gamma} ds$.

[1] [Ahl78, p.102] proof of this is as follows. The first claim is that for any continuous function $g : [a,b] \to \mathbb{C}$, we have

$$\left|\int_a^b g(t)dt\right| \leq \int_a^b |g(t)|\, dt.$$

Indeed, for any real $\theta$, we have

$$\text{Re}\left[e^{-i\theta}\int_a^b g(t)\, dt\right] = \int_a^b \text{Re}\left[e^{-i\theta}g(t)\right]\, dt \leq \int_a^b |g(t)|\, dt.$$

Then taking any $\theta$ such that $\int_a^b g(t)\, dt = re^{i\theta}$, this gives the claim. Then take $g(t) = f(\gamma(t))\gamma'(t)$ to obtain the result (88) above.

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**Lemma 177.2.9.** *Let $U \subseteq \mathbb{C}$ be an open subset. For a continuous function $f : U \to \mathbb{C}$, the following are equivalent:*

(1) *$f$ is holomorphic,*
(2) *for every $z_0 \in U$ and every open disc $B_\epsilon \subseteq U$ containing $z_0$ with $\overline{B}_\epsilon \subseteq U$, we have*

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$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B_\epsilon} \frac{f(z)}{z - z_0}dz, \tag{89}$$

(3) *$f$ is complex analytic.*

PROOF. We sketch the proof. Suppose (1) first, that $f$ is holomorphic. To prove (2), the key point is that the function $f(z)/(z - z_0)$ is holomorphic everywhere in $B_\epsilon$ except for the point $z_0$. Therefore, using say Stoke's Theorem, the integral in (89) is the same for every positively oriented circle $C_r := \partial B_r$ of positive radius $r$ contained in the disk $B_\epsilon$, and containing $z_0$ in its interior. Now let us focus on such circles centered at $z_0$, and consider:

$$\begin{aligned}\int_{C_r} \frac{f(z)}{(z - z_0)}dz &= \int_{C_r} \frac{f(z_0)}{(z - z_0)}dz + \int_{C_r} \frac{f(z) - f(z_0)}{(z - z_0)}dz \\ &= 2\pi i f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{(z - z_0)}dz.\end{aligned}$$

Using the bound on the modulus of the integral (88), and taking the limit as $r$ goes to 0, the integral on the right goes to 0, and one obtains (89).

We now show (2) implies (3). For this, we will use a special case of [**Rud87**, Thm 10.7, p.199], and get analyticity. The point is to show that the function

$$g(w) := \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - w}dz$$

is an analytic function in $w$ on $B_\epsilon(z_0)$. The point is that by assumption of (2) we have $g(w) = f(w)$. Note that holomorphicity is immediate, using that $1/(z - w)$ has continuous partial differentials (in $w$), to pass $\frac{\partial}{\partial \overline{w}}$ through the integral, and use that $\frac{\partial}{\partial \overline{w}}\frac{1}{z-w} = 0$ since $z \neq w$. To prove analyticity, we use that for $w \in B_\epsilon(z_0)$, we have

$$\frac{1}{z - w} = \frac{1}{z - z_0}\frac{1}{1 - \frac{w - z_0}{z - z_0}} = \frac{1}{z - z_0}\sum_{n=0}^{\infty}\frac{(w - z_0)^n}{(z - z_0)^n} = \sum_{n=0}^{\infty}\frac{(w - z_0)^n}{(z - z_0)^{n+1}}$$

converges uniformly in $z$ for fixed $w \in B_\epsilon(z_0)$. Therefore, we have $\frac{f(z)}{(z-w)} = \sum_{n=0}^{\infty} f(z)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}$, which, since $f$ is continuous, can be shown to be uniformly convergent in $z$ for fixed $w \in B_\epsilon(z_0)$. Then integrating against $dz$, using uniform convergence, and that $w - z_0$ is constant with respect to $z$, we see that $g(w) = \sum_{n=0}^{\infty} a_n(w - z_0)^n$ where $a_n = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}dz$. ☐

REMARK 177.2.10. The proof above shows that if $f : U \to \mathbb{C}$ is just assumed to be continuous, then

$$f(z_0) = \lim_{\epsilon \to 0} \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0}dz.$$

**2.1. Basic properties of holomorphic maps.** Here we review a few basic facts about holomorphic maps.

2.1.1. *Inverse function theorem.*

**Theorem 177.2.11** (Inverse function)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic map. If $f'(z_0) \neq 0$, then $f$ is locally a holomorphic isomorphism near $z_0$.*

PROOF. Use the real inverse function theorem, and the fact that the inverse of a conformal matrix is conformal. □

REMARK 177.2.12. Using the local structure theorem below (Theorem 177.2.13), one can show that if a holomorphic function is injective, then it is biholomorphic onto its image (indeed, locally it must be $z \mapsto z$, so it is locally biholomorphic; but it is a bijection so it is globally biholomorhpic). The same result will hold for maps $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$, but may fail if the dimensions of the source and target are not the same; e.g., $z \mapsto (z^3, z^2)$ is holomorphic and injective, but not biholomorphic onto its image.

2.1.2. *Local structure theorem.*

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**Theorem 177.2.13** (Local structure theorem)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic map with $f(0) = 0$. Then locally, $f$ factors as a holomorphic isomorphism followed by $z \mapsto z^m$, followed by a holomorphic isomorphism.*

REMARK 177.2.14. More precisely we mean that for each point $p \in U$, there is an open neighborhood $p \in U_p \subseteq U$, and an open ball $B_\epsilon(0)$, so that $f|_{U_p} : B_\epsilon(p) \to f(U_p)$ factors as

$$U_p \xrightarrow[\sim]{f_1} B_\epsilon(0) \xrightarrow{z \mapsto z^m} B_{\epsilon^m}(0) \xrightarrow[\sim]{f_2} f(U_p)$$

where $f_1$ and $f_2$ are holomorphic isomorphisms.

PROOF. To make the formulas simpler, we may as well take $p = 0$ and $f(0) = 0$. We have immediately from analyticity that $f(z) = z^m h(z)$, where $h(z)$ is nowhere vanishing in a neighborhood of 0. We claim there is $g(z)$ such that $g(z)^m = h(z)$. In short, using that $h(z)$ is nowhere vanishing, and possibly taking a smaller neighborhood, we can define a branch of log and set $g(z) = \exp(\frac{1}{m} \log h(z))$. Thus $f(z) = (zg(z))^m$. Then we set $f_1(z) = zg(z)$, and $f_2(z) = z$. We can see that $f_1(z)$ is locally a holomorphic isomorphism by considering $f_1'(z) = g(z) + zg'(z)$, so that $f_1'(0) = g(0) \neq 0$. Then we use the inverse function theorem. □

REMARK 177.2.15. To avoid technicalities with logs, just observe that $h'(z)/h(z)$ is holomorphic near 0. Therefore, using analyticity of holomorphic functions, we can find $a(z)$ such that $a'(z) = h'(z)/h(z)$. Then we have $\frac{d}{dz}(h(z)e^{-a(z)}) = 0$, so that $h(z) = Ce^{a(z)}$ for some constant $C$. Then we set $g(z) = e^{\frac{C}{m}a(z)}$.

REMARK 177.2.16. The number $m$ is determined uniquely at a point $p \in U$ by the number of preimages of $f$ near $f(p)$, or equivalently, by the order of vanishing of $f$ at $p$.

**Corollary 177.2.17.** *The zero set of a nonconstant holomorphic function has no limit points in the domain of definition. In particular, the zero set does not contain any open subsets.*

PROOF. This follows immediately from the structure theorem (and the elementary case of $z \mapsto z^m$). □

REMARK 177.2.18 (Limit points). Recall that a *limit point* of a subset $S$ of a topological space $X$ is a point $x \in X$ so that every open neighborhood of $x$ contains a point of $S$ *other than* $x$. An isolated point of $S$ is a point $s \in S$ such that there exists on open neighborhood of $s$ whose intersection with $S$ contains only the point $s$. Since the closure of $S$ is the set of points $x$ in $X$ so that every open neighborhood of $x$ meets $S$, we see that the limit points of $S$ are exactly the elements of the closure of $S$ that are not also isolated points of $S$. In other words, $\overline{S} = L(S) \sqcup I(S)$, where $L(S)$ (resp. $I(S)$) is the set of limit (resp. isolated) points of $S$.

The following is a useful fact: if there is no point $x \in X$ such that $\{x\}$ is open, then if $S$ has no limit points in $X$, then $S$ is nowhere dense (i.e., the closure has empty interior). Indeed, if $S$ has no limit points, then $S = \overline{S} = I(S)$. Let $s \in S = I(S)$; the claim is that $s$ cannot be in the interior of $S$. To this end, let $U_s \subseteq X$ be an open neighborhood of $s$ such that $U_s \cap S = \{s\}$. If there were an open subset $V$ of $X$ with $V \subseteq S$ that also contained $s$, then $V \cap U_s = \{s\}$ would be an open subset of $X$, contradicting our assumption on the topology of $X$. Note that the converse fails; even if there is no point $x \in X$ such that $\{x\}$ is open, there can be sets $S$ that are nowhere dense that do have limit points. Indeed, in $U = B_1(0) \subseteq \mathbb{C}$, take $S = \{1/n : n \geq 2\}$. This set has closure with empty interior, but it has a limit point, 0.

2.1.3. *Open mapping theorem.*

**Theorem 177.2.19** (Open mapping)**.** *Nonconstant holomorphic maps are open (take open sets to open sets).*

PROOF. This follows immediately from the local structure theorem. □

2.1.4. *Maximum principle.*

**Theorem 177.2.20** (Maximum principle)**.** *Let $U \subseteq \mathbb{C}$ be open and connected. If $f : U \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no local maximum in $U$. If $U$ is bounded and $f$ can be extended to a continuous function $f : \overline{U} \to \mathbb{C}$, then $|f|$ takes its maximal values on the boundary $\partial U$.*

PROOF. Use the open mapping theorem. □

2.1.5. *Identity theorem.*

**Theorem 177.2.21** (Identity theorem)**.** *If $f, g : U \to \mathbb{C}$ are two holomorphic functions on a connected open subset $U \subseteq \mathbb{C}$ such that $f(z) = g(z)$ for all $z$ in a non-empty open subset $V \subseteq U$, then $f = g$.*

REMARK 177.2.22. There are stronger versions of the identity theorem (e.g., take any subset $V$ with limit points), but in this form it immediately generalizes to higher dimensions.

PROOF. From the corollary to the local structure theorem we have that zero sets of nonconstant holomorphic functions have no limit points (in the domain of definition). To prove the identity theorem, take the difference of the two functions and consider the zero set. □

2.1.6. *Riemann extension theorem.*

**Theorem 177.2.23** (Riemann extension theorem)**.** *Let $f : B_\epsilon(z_0)^* \to \mathbb{C}$ be a bounded holomorphic function on a punctured disk. Then $f$ can be extended to a holomorphic function $f : B_\epsilon(z_0) \to \mathbb{C}$.*

PROOF. The boundedness shows that $g(z) = (z - z_0)^2 f(z)$ is complex differentiable at $z_0$, and therefore given by a power series. The claim is that $g(z)$ vanishes to at least order 2 at $z_0$; otherwise, since $g(z)$ is continuous, one would have that $f(z) = g(z)/(z - z_0)^2$ was not be bounded (show $1/z^2$ is not bounded). Consequently, dividing the power series for $g(z)$ by $(z - z_0)^2$, we have that $f(z)$ is analytic. □

2.1.7. *Riemann mapping theorem.*

**Theorem 177.2.24** (Riemann mapping theorem)**.** *Let $U \subsetneq \mathbb{C}$ be a simply connected open subset properly contained in $\mathbb{C}$. Then $U$ is biholomorphic to the unit ball $B_1(0)$; i.e., there exists a bijective holomorphic map $f : U \to B_1(0)$ such that its inverse $f^{-1}$ is also holomorphic.*

PROOF. We refer the reader to [Rud87, Thm. 14.8, p.283]. □

2.1.8. *Liouville's theorem.*

**Theorem 177.2.25** (Liouville's Theorem)**.** *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant. In particular, there is no biholomorphic map between $\mathbb{C}$ and a ball $B_\epsilon(0)$ with $\epsilon < \infty$.*

PROOF. Using analyticity, it is not hard to show that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Then on a circle $C_R$ of radius $R$ centered about $z_0$, if $|f(z)| \le M_R$ for all $z \in C_R$, then the derivatives of $f$ at $z_0$ satisfy (for each $n \in \mathbb{N}$)

$$\left| f^{(n)}(z_0) \right| \le \frac{n! M_R}{R^n}.$$

Apply this to the first derivative, and let $R \to \infty$. □

2.1.9. *Residue theorem.*

**Theorem 177.2.26** (Residue theorem)**.** *Let $f : B_\epsilon(z_0)^* \to \mathbb{C}$ be a holomorphic function on a punctured disk. Then $f$ can be expanded in a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and the coefficient $a_{-1}$ is given by the residue formula*

$$a_{-1} = \frac{1}{2\pi i} \int_{\partial B_{\epsilon/2}(z_0)} f(z) dz.$$

PROOF. The existence of the Laurent series is very similar to the proof of Lemma 177.2.9. The starting point is to show that we have

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $z$ sits inside the first circle, and outside of the second circle (both centered at $z_0$). This follows from Lemma 177.2.9 once one cuts the annulus defined by the two circles into a pacman containing $z$ and a wedge not containing $z$ (the integral around the wedge gives zero, and the lemma applies to the pacman).

Then since $z$ sits inside the first circle, the analysis in Lemma 177.2.9 essentially gives that the first integral is given by a power series. Replacing $z$ by $1/z$, one inverts the circles so to speak, and then the same analysis gives that the second integral is given by a power series in $1/z$.

To get the value $a_{-1}$, we just integrate and use uniform convergence. □

2.1.10. *Schwarz lemma.*

**Lemma 177.2.27** (Schwarz Lemma)**.** *Let $f$ be a holomorphic function on an open neighborhood of the closure of the disk $B\epsilon(0)$. Assume that $f$ vanishes to order $k$ at the origin. If there is some real number $C$ such that $|f(z)| \leq C$ for all $z \in \overline{B_\epsilon(0)}$, then actually there is the possibly stronger bound:*

$$|f(z)| \leq C\left(\frac{|z|}{\epsilon}\right)^k$$

*for all $z \in \overline{B_\epsilon(0)}$.*

REMARK 177.2.28. In short, we know the maximum, say $C$, of $|f(z)|$ occurs on the boundary circle $C_\epsilon(0)$. However, we can estimate how much smaller the modulus of $f$ is on the interior, by multiplying $C$ by the fraction of the distance we are to the boundary circle (to the power $k$).

PROOF. Fix $z \in B_\epsilon(0)$ and define a holomorphic function $g_z(w)$ on the open neighborhood of the closure of the disk $B\epsilon(0)$ on which $f$ is defined, as follows: For such $w$, one sets

$$g_z(w) := w^{-k} f\left(w \cdot \frac{z}{|z|}\right).$$

Then for $|w| = \epsilon$ we have $|g_z(w)| \leq \epsilon^{-k}C$. The maximum principle then implies the same bound $g_z(w)| \leq \epsilon^{-k}C$ for all $|w| \leq \epsilon$. Hence

$$|z|^{-k}|f(z)| = |g_z(|z|)| \leq \epsilon^{-k}C,$$

giving the desired bound. □

## 3. Meromorphic functions

Let $U \subseteq \mathbb{C}$ be open. Informally, meromorphic function $f$ on $U$ is a ratio $f = g/h$ of holomorphic functions on $U$, up to the obvious equivalence. A little more precisely, it is an element of the fraction field of the integral domain of holomorphic functions on $U$. When we start thinking about complex manifolds, we will have to work a little harder, since we will not have enough global holomorphic functions to define things this way. Observing that for $U \subseteq \mathbb{C}$, multiplying by products of powers of $z - p$ for different points $p$ (or more generally Ahlfors Theorem 7 p.195, and generalizing the proof to arbitrary open sets), one can define meromorphic functions equivalently to be functions given locally by the ratio of holomorphic functions. To make this geometric, we make the following definition.

Given a nowhere dense (i.e., closure has empty interior) subset $S \subseteq U$ (e.g., $S$ has no limit points in $U$), and a map of sets $f : U - S \to \mathbb{C}$, we say $(S, f)$ is a representative for a meromorphic function on $U$ if there exist:

- an open cover $U = \bigcup_{i \in I} U_i$,

- holomorphic functions $g_i, h_i : U_i \to \mathbb{C}$,

satisfying

$$h_i|_{U_i - S} \cdot f|_{U_i - S} = g_i|_{U_i - S}$$

for every $i$. We say that $(S, f) \sim (S', f')$ if setting $S'' = S \cup S'$, then $f|_{U - S''} = f'|_{U - S''}$. A meromorphic function on $U$ is an equivalence class of representatives.

REMARK 177.3.29. One can show that the set of meromorphic functions on $U$ is a field.