Hodge Theory, Almost Complex Structures, and the Lefschetz Decomposition

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Contents

Introduction

Cohomology of compact Kähler manifold has a Hodge decomposition. This motivates the definition of an abstract Hodge structure. Useful to study abstract Hodge structures algebraically. The material can be found in [Huy05] and [PS08]

(1) Motivation
(2) Main topics
(3) Outline

1. The Category of Hodge Structures

Definition 1.1 (Hodge structure)**.** A Hodge structure is a pair consisting of a finite dimensional $\mathbb{R}$-vector space V with a decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

where $\overline{V^{p,q}} = V^{q,p}$.

From now on we will refer to a Hodge structure as V without explicitly mentioning the decomposition into direct summands $V^{p,q}$. We define morphisms of Hodge structures as follows:

Definition 1.2 (Morphism of Hodge structures)**.** Given Hodge structures V and W, a morphism from V to W is a linear map $\phi : V \to W$ such that $\phi(V^{p,q}) \subseteq W^{p,q}$.

We can view the collection of all Hodge structures as the objects of a category which will be denoted $\mathbf{H}$. Given an integer k, a Hodge structure V with $V^{p,q} = 0$ unless $p + q = k$ is called a weight k Hodge structure. We denote by $\mathbf{H}_k$ the category of weight-k Hodge structures. If we further have that $k \geq 0$ and $V^{p,q} = 0$ whenever $p < 0$ or $q < 0$ then V is a pure weight k Hodge structure. We denote by $\mathbf{pH}_k$ the category of pure weight-k Hodge structures.

Lemma 1.3. *Let V, W and X be finite dimensional $\mathbb{R}$-vector spaces. Then* $(V \oplus W) \otimes_{\mathbb{R}} X = (V \otimes_{\mathbb{R}} X) \oplus (W \otimes_{\mathbb{R}} X)$

Proof. First we construct a bilinear map

$$\Phi : (V \oplus W) \times X \to (V \otimes_{\mathbb{R}} X) \oplus (W \otimes_{\mathbb{R}} X)$$

which is given by

$$\Phi(v, w, x) = (v \otimes x, w \otimes x)$$

This map is bilinear because given any $(v_1, w_1), (v_2, w_2) \in V \oplus W$ and any $x \in X$ we have

$$\Phi(v_1+v_2, w_1+w_2, x) = ((v_1+v_2)\otimes x, (w_1+w_2)\otimes x) = (v_1\otimes x+v_2\otimes x, w_1\otimes x+w_2\otimes x) =$$

$$(v_1 \otimes x, w_1 \otimes x) + (v_2 \otimes x, w_2 \otimes x) = \Phi(v_1, w_1, x) + \Phi(v_2, w_2, x)$$

which shows bilinearity in the first argument. Also for $(v, w) \in V \oplus W$ and $x_1, x_2 \in \mathbb{C}$ we have

$$\Phi(v, w, x_1+x_2) = (v\otimes(x_1+x_2), w\otimes(x_1+x_2)) = (v\otimes x_1+v\otimes x_2, w\otimes x_1+w\otimes x_2)$$

$$= (v \otimes x_1, w \otimes x_1) + (v \otimes x_2, w \otimes x_2) = \Phi(v, w, x_1) + \Phi(v, w, x_2)$$

which shows bilinearity in the second argument. Then by the universal property of the tensor product there is a unique linear map

$$\Psi : (V \oplus W) \times X \to (V \otimes_{\mathbb{R}} X) \oplus (W \otimes_{\mathbb{R}} X)$$

such that the following diagram commutes.

$$\begin{tikzcd}
(V \oplus W) \times X \arrow[r] \arrow[d] & (V \otimes_{\mathbb{R}} X) \oplus (W \otimes_{\mathbb{R}} X) \\
(V \oplus W) \otimes_{\mathbb{R}} X \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Suppose we are given an arbitrary $\sum_{i=1}^{k}(v_i + w_i) \otimes x_i$ such that

$$\Psi(\sum_{i=1}^{k}(v_i + w_i) \otimes x_i) = 0$$

then it follows that

$$\sum_{i=1}^{k}(v_i \otimes x_i) + (w_i \otimes x_i) = 0$$

so that we have $\sum_{i=1}^{k}(v_i \otimes x_i) = 0$ and $\sum_{i=1}^{k}(w_i \otimes x_i) = 0$. Therefore

$$\sum_{i=1}^{k}(v_i \otimes x_i) + \sum_{i=1}^{k}(w_i \otimes x_i) = \sum_{i=1}^{k}(v_i + w_i) \otimes x_i = 0$$

which shows that Ψ is injective.

Now we claim that $(V \oplus W) \otimes_{\mathbb{R}} X$ and $(V \otimes_{\mathbb{R}} X) \oplus (W \otimes_{\mathbb{R}} X)$ have the same dimension as $\mathbb{R}$-vector spaces. Suppose V, W and X have dimensions n, m and p respectively. Then $V \oplus W$ is dimension $n + m$ hence $(V \oplus W) \otimes_{\mathbb{R}} X$ is dimension $(n + m)p$ over $\mathbb{R}$. On the other hand $V \otimes_{\mathbb{R}} X$ and $W \otimes_{\mathbb{R}} X$ have dimensions np and mp respectively. But then $(V \otimes_{\mathbb{R}} X) \oplus (W \otimes_{\mathbb{R}} X)$ is dimension $np + mp$ which completes the proof of the claim.

Since Ψ is an injective linear map between vector spaces of the same dimension it must also be surjective by the rank-nullity theorem. This shows that Ψ is an isomorphism which completes the proof. $\square$

Corollary 1.4. *If V and W are finite dimensional $\mathbb{R}$ vector spaces then*

$$(V \oplus W)_{\mathbb{C}} = V_{\mathbb{C}} \oplus W_{\mathbb{C}}$$

Proof. This is just the above lemma with $X = \mathbb{C}$ $\square$

Proposition 1.5. *If V and W are Hodge structures then $V \oplus W$ has a natural Hodge structure.*

Proof. We need to find a Hodge decomposition for $(V \oplus W)_{\mathbb{C}} = (V \oplus W) \otimes \mathbb{C}$. Using the corollary to lemma 1.3 we have

$$(V \oplus W)_{\mathbb{C}} = V_{\mathbb{C}} \oplus W_{\mathbb{C}} = \left(\bigoplus_{p+q=k} V^{p,q}\right) \oplus \left(\bigoplus_{r+s=k} W^{r,s}\right)$$

To find a Hodge decomposition for $(V \oplus W)_{\mathbb{C}}$ set

$$(V \oplus W)^{i,j} = V^{i,j} \oplus W^{i,j}$$

From the above it is clear that $(V \oplus W)_{\mathbb{C}} = \bigoplus_{i+j=k'} (V \oplus W)^{i,j}$. We also have

$$\overline{(V \oplus W)^{i,j}} = \overline{V^{i,j} \oplus W^{i,j}} = \overline{V^{i,j}} \oplus \overline{W^{i,j}} = V^{j,i} \oplus W^{j,i} = (V \oplus W)^{j,i}$$

□

Corollary 1.6 (Products and coproducts exist)**.** *Products and coproducts exist in the category of Hodge structures (resp. weight-k Hodge structures, resp. pure weight-k Hodge structures).*

Proof. Consider the above Hodge structure on $V \oplus W$ and let $\phi_1 : V \to P$ and $\phi_2 : W \to P$ be morphisms of Hodge structures. By the universal property of $V \oplus W$ there is a unique map $\rho : V \oplus W \to P$ such that $\phi_1 = \rho \circ i_V$ and $\phi_2 = \rho \circ i_W$. Clearly the inclusion maps ι_V and ι_W are morphisms of Hodge structures. If $v + w \in (V \oplus W)^{i,j}$ then

$$\rho(v + w) = \rho(v) + \rho(w) = (\rho \circ i_V)(v) + (\rho \circ i_W)(w) =$$

$$\phi_1(v) + \phi_2(w) \in V^{i,j} \oplus W^{i,j} = (V \oplus W)^{i,j}$$

where the last step follows because the maps ϕ_1 and ϕ_2 are morphisms of Hodge structures. But then ρ must also be a morphism of Hodge structures and this shows that the category of Hodge structures has coproducts. A similar argument with the inclusion maps replaced by projection maps $\pi_V : V \oplus W \to V$ and $\pi_W : V \oplus W \to W$ gives products in $\mathbf{H}$. The same proof works for $\mathbf{H}_k$ and $\mathbf{pH}_k$. □

Also, the tensor product of pure weight k and k' Hodge structures is weight $k+k'$ as we will now show.

Lemma 1.7.

(1) *If V and W are $\mathbb{R}$-vector spaces then* $(V \otimes_{\mathbb{R}} W)_{\mathbb{C}} = V_{\mathbb{C}} \otimes_{\mathbb{R}} W_{\mathbb{C}}$

(2) *if V_1, V_2, ...V_n and W_1, W_2, ... W_m are $\mathbb{R}$-vector spaces then*

$$(V_1 \oplus \cdots \oplus V_n) \otimes (W_1 \oplus \cdots \oplus W_m) = \bigoplus_{p,q} V_p \otimes W_q$$

Proof. For (1) Note that

$$V_{\mathbb{C}} \otimes_{\mathbb{R}} W_{\mathbb{C}} = (V \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} (W \otimes_{\mathbb{R}} \mathbb{C})$$

hence from the commutativity and associativity of the tensor product we have

$$(V \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} (W \otimes_{\mathbb{R}} \mathbb{C}) = V \otimes_{\mathbb{R}} W \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} =$$

$$V \otimes_{\mathbb{R}} W \otimes_{\mathbb{R}} \mathbb{C} = (V \otimes_{\mathbb{R}} W)_{\mathbb{C}}$$

where the second to last equality holds because $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}$.

For (2) we note that lemma 1.3 gives $(V_1 \oplus V_2) \otimes W_1 = (V_1 \otimes W_1) \oplus (V_2 \otimes W_1)$ Then we induct on n to show that

$$(V_1 \oplus V_2 \cdots \oplus V_n) \otimes W_1 = (V_1 \otimes W_1) \oplus \cdots \oplus (V_n \otimes W_1)$$

This is trivially true when $n = 1$ and it is true for $n = 2$ by the above. Hence suppose the statement is true for arbitrary $n-1$ with $n > 2$ and note that

$$(V_1 \oplus V_2 \cdots \oplus V_n) \otimes W_1 = ((V_1 \oplus V_2 \cdots \oplus V_{n-1}) \oplus V_n) \otimes W_1 =$$

$$((V_1 \oplus V_2 \cdots \oplus V_{n-1}) \otimes W_1) \oplus (V_n \otimes W_1) = ((V_1 \otimes W_1) \oplus \cdots \oplus (V_{n-1} \otimes W_1)) \oplus (V_n \otimes W_1) =$$

$$(V_1 \otimes W_1) \oplus \cdots \oplus (V_n \otimes W_1)$$

Setting $V = V_1 \oplus \cdots \oplus V_n$ we can induct on m using the exact same argument above to see that

$$V \otimes (W_1 \oplus \cdots \oplus W_m) = (V \otimes W_1) \oplus \cdots \oplus (V \otimes W_m)$$

but then it follows that

$$(V_1 \oplus \cdots \oplus V_n) \otimes (W_1 \oplus \cdots \oplus W_m) = \left(\bigoplus_{i=1}^{n} V_i\right) \otimes W_1 \oplus \cdots \oplus \left(\bigoplus_{i=1}^{n} V_i\right) \otimes W_m =$$

$$\bigoplus_{p,q} V_p \otimes W_q$$

□

1.1. **The category of Hodge structures is abelian.** The category **H** turns out to be an abelian category. Recall that a category **C** is Abelian if

(1) For any objects A, B the set $\mathrm{Hom}(V, W)$ has an abelian group structure where the group operation $+$ is bilinear with respect to function composition
(2) For any objects $A_1, A_2, ..., A_n$ of **C** there exists a product and coproduct, and these universal objects coincide.
(3) kernels and cokernels exist for all the morphisms of **C**
(4) f is a monomorphism iff it is a kernel and is an epimorphism iff it is a cokernel

Proposition 1.8. *The category of Hodge structures* **H** *is an abelian category and for every $k \in \mathbb{Z}$ the category of weight k Hodge Structures* $\mathbf{H}_k$ *is an abelian subcategory of* **H**

Proof. For the first part we will prove each of the items on the list above holds in the category **H**. It is helpful to notice that as defined **H** is a subcategory of **V** the category of vector spaces. Therefore to prove the proposition we can verify that the required universal objects and morphisms of $\mathbf{V}$ are also a part of $\mathbf{H}$

(1) We will show that $\mathrm{Hom}_{\mathbf{H}}(V,W)$ is a subgroup of $\mathrm{Hom}_{\mathbf{V}}(V,W)$. Suppose $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathbf{H}}(V,W)$ then $\phi_1 - \phi_2$ is a linear map from V to W and we must show that $\phi_1 - \phi_2(V^{i,j}) \subseteq W^{i,j}$. For $v \in V^{i,j}$ we have $\phi_1(v), \phi_2(v) \in W^{i,j}$ hence $\phi_1(v) - \phi_2(v) \in W^{i,j}$ since $W^{i,j}$ is a subspace. This shows that $\phi_1 - \phi_2 \in \mathrm{Hom}_{\mathbf{H}}(V,W)$ which shows that $\mathrm{Hom}_{\mathbf{H}}(V,W) \leq \mathrm{Hom}_{\mathbf{V}}(V,W)$. Furthermore let $\phi_1, \phi_2 \in \mathrm{Hom}(V,V')$ and $\rho \in \mathrm{Hom}(V',W)$. Then given $v \in V$ we have

$$\rho \circ (\phi_1 + \phi_2)(v) = \rho(\phi_1(v) + \phi_2(v)) = (\rho \circ \phi_1)(v) + (\rho \circ \phi_2)(v)$$

The proof of bilinearity in the other argument of $\circ$ is analogous.

(2) We will use induction on n to prove that $V_1 \oplus \cdots \oplus V_n$ has a Hodge structure. By proposition 1.5 the statement is true for $n = 2$. Suppose it is true for all sets of n objects with $n > 2$. Given $V_1, ...V_n, V_{n+1}$ objects of $\mathbf{H}$ by the induction hypothesis there is a Hodge decomposition of $(V_1 \oplus ... \oplus V_n)_{\mathbb{C}}$. Then using 1.5 again there is a Hodge decomposition of $((V_1 \oplus ... \oplus V_n) \oplus V_{n+1})_{\mathbb{C}} = (V_1 \oplus ... \oplus V_n \oplus V_{n+1})_{\mathbb{C}}$. Next suppose we are given arbitrary $k \in \{1, ..., n\}$ and arbitrary $v \in V_k^{i,j}$. Then we have

$$\iota_k(v) \in V_k^{i,j} \subset V_1^{i,j} \oplus \cdots \oplus V_n^{i,j}$$

where $\iota_k : V_k \to V_1 \oplus \cdots \oplus V_n$ is an inclusion map, namely the universal morphism from V_k into the coproduct $V_1 \oplus \cdots \oplus V_n$. But then the above says ι_k is a morphism of hodge structures for arbitrary k. A similar argument shows that the projections, i.e. universal morphisms corresponding to the direct product $\pi_k : V_1 \oplus \cdots \oplus V_n \to V_k$ is a morphism of hodge structures. This shows that the category of Hodge structures has products and coproducts.

(3) Let $\phi : V \to W$ be a morphism of Hodge structures. We need to show that $\ker(\phi)$ has a Hodge structure. Consider $\ker(\phi) \otimes \mathbb{C} = \ker(\phi)_{\mathbb{C}}$ and notice that it is equal to $\ker(\phi_{\mathbb{C}})$ where $\phi_{\mathbb{C}}$ is the map ϕ extended $\mathbb{C}$-linearly to $V_{\mathbb{C}}$. Explicitly define $\phi_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ to be the map given on simple tensors by

$$v \otimes z \mapsto \phi(v) \otimes z$$

We will show that defining $\ker(\phi)^{i,j} = \ker(\phi_{\mathbb{C}}) \cap V^{i,j}$ gives a Hodge decomposition of $\ker(\phi)_{\mathbb{C}}$. If $(p, q) \neq (r, s)$ since $V_{\mathbb{C}}$ is a direct sum of the subspaces $V^{i,j}$ we have $V^{p,q} \cap V^{r,s} = \{0\}$ so that

$$\ker(\phi)^{p,q} \cap \ker(\phi)^{r,s} = \ker(\phi_{\mathbb{C}}) \cap V^{i,j} \cap V^{r,s} = \ker(\phi)_{\mathbb{C}} \cap \{0\} = \{0\}$$

Furthermore given any $v \in \ker(\phi)_{\mathbb{C}} \subseteq V_{\mathbb{C}}$ we have $v = \Sigma_{i,j} v_{i,j}$ hence $\phi(v) = \phi(\Sigma_{i,j} v_{i,j}) = \Sigma_{i,j} \phi(v_{i,j}) = 0$. For each pair (i, j) we must have $\phi_{\mathbb{C}}(v_{i,j}) \in W^{i,j}$ because ϕ is a morphism of Hodge structures. Since $W_{\mathbb{C}} = \bigoplus_{i,j} W^{i,j}$ the equality $\Sigma_{i,j} \phi(v_{i,j}) = 0$ implies that $\phi(v_{i,j}) = 0$ for each (i, j). This shows that $v_{i,j} \in \ker(\phi)$ for each (i, j) hence $v \in \bigoplus_{i,j} \ker(\phi)^{i,j}$. Combining this with the above observation gives

$$\ker(\phi) = \bigoplus_{i,j} \ker(\phi)^{i,j}$$

To verify the conjugacy requirement note that we have

$$\overline{\ker(\phi)^{i,j}} = \overline{\ker(\phi_{\mathbb{C}}) \cap V^{i,j}} = \overline{\ker(\phi_{\mathbb{C}})} \cap \overline{V^{i,j}} =$$

$$\overline{\ker(\phi_{\mathbb{C}})} \cap V^{j,i} = \ker(\phi_{\mathbb{C}}) \cap V^{j,i} = \ker(\phi)^{j,i}$$

It remains to show that $\overline{\ker(\phi_{\mathbb{C}})} = \ker(\phi_{\mathbb{C}})$ which is easy when using the fact that $\ker(\phi)_{\mathbb{C}} = \ker(\phi) \otimes \mathbb{C}$ and $\overline{\ker(\phi) \otimes \mathbb{C}} = \overline{\ker(\phi)} \otimes \overline{\mathbb{C}} = \ker(\phi) \otimes \mathbb{C}$. A technical point is that in this case we are viewing $V \subset V_{\mathbb{C}}$ and $W \subset W_{\mathbb{C}}$ under the inclusion $v \mapsto v \otimes 1$ (respectively $w \mapsto w \otimes 1$). Then in this context ϕ is the map $V \to W$ defined by $v \otimes 1 \mapsto \phi(v) \otimes 1$. To complete the proof we note that since $\ker(\phi)^{i,j} = \ker(\phi_{\mathbb{C}}) \cap V^{i,j}$ the inclusion map is $\iota : \ker(\phi) \to V$ is a morphism of Hodge structures.

Next we wish to show that the category **H** has cokernels. To do this suppose we are given W and V with $W \leq V$. To simplify notation define $W^{i,j} = W_{\mathbb{C}} \cap V^{i,j}$, then we will find a Hodge decomposition for $\frac{V}{W}_{\mathbb{C}}$. Define a map $\phi : \bigoplus_{i,j} V^{i,j} \to \bigoplus_{i,j} \frac{V^{i,j}}{W^{i,j}}$ by

$$\phi(\Sigma_{i,j} v_{i,j}) = \Sigma_{i,j}[v_{i,j}]$$

where $[v_{i,j}]$ denotes the conjugacy class containing $v_{i,j}$. This map is clearly surjective so suppose that we have $v = \Sigma_{i,j} v_{i,j} \in \ker(\phi)$. Then $\Sigma_{i,j}[v_{i,j}] = 0$ in $\bigoplus_{i,j} \frac{V^{i,j}}{W^{i,j}}$ which gives $[v_{i,j}] = 0$ for each pair (i,j) so that $v_{i,j} \in W^{i,j}$. This shows that $\ker(\phi) = \bigoplus_{i,j} W^{i,j}$ so by the first isomorphism theorem we have

$$\frac{V_{\mathbb{C}}}{W_{\mathbb{C}}} = \frac{\bigoplus_{i,j} V^{i,j}}{\bigoplus_{i,j} W^{i,j}} \cong \bigoplus_{i,j} \frac{V^{i,j}}{W^{i,j}}$$

The conjugacy requirement for this decomposition follows from the conjugacy requirement for the decomposition of $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$. Since $\frac{V}{W}_{\mathbb{C}} \cong \frac{V_{\mathbb{C}}}{W_{\mathbb{C}}}$ this implies that **H** has cokernels. Lastly the fourth requirement holds because morphisms of Hodge structures are morphisms in the abelian category of vector spaces.

□

Proposition 1.9. *If V and W are Hodge structures then $V \otimes W$ has a natural Hodge structure. If V and W have weights k and k' respectively then $V \otimes W$ has weight $k + k'$*

Proof. From the lemma $(V \otimes W)_{\mathbb{C}} = \oplus_{i,j} V^{i,j} \otimes \oplus_{p,q} W^{p,q} = \oplus_{r,s}(V \otimes W)^{r,s}$ where $(V \otimes W)^{r,s} = \bigoplus_{i+p=r, j+q=s} V^{i,j} \otimes W^{p,q}$. The conjugation requirement holds because for any direct summand $V^{i,j} \otimes W^{p,q}$ in $(V \otimes W)^{r,s}$ we have

$$\overline{V^{i,j} \otimes W^{p,q}} = \overline{V^{i,j}} \otimes \overline{W^{p,q}} = V^{j,i} \otimes W^{q,p}$$

and the latter term is a direct summand of the product defining $(V \otimes W)^{s,r}$. Finally let V and W have weights k and k' respectively and suppose that (r, s) is such that $r+s \neq k+k'$. Then given any $V^{i,j} \otimes W^{p,q}$ with $i+j+p+q = r+s$ we have either $i+j \neq r$ or $p+q \neq s$. In other words either $W^{i,j} = \{0\}$ or $V^{p,q} = \{0\}$ and in both of these cases we have $V^{i,j} \otimes W^{p,q} = \{0\}$. This shows that $(V \otimes W)^{r,s} = \{0\}$ whenever $r + s \neq k + k'$ so that $V \otimes W$ is a weight $k + k'$ hodge structure. □

2. Hodge Structures via Representation Theory

Another way to define a Hodge structure is by using representation theory. In this section we will view $\mathbb{C}^*$ as a real algebraic group.

$$\mathbb{C}^* = \{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 + b^2 \neq 0\}.$$

Recall that a a representation $\phi : \mathbb{C}^* \to GL(\mathbb{R}^n)$ is called *algebraic* if ϕ is given by polynomials in a, b; i.e., for $z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{C}^*$ if we write $\phi(z)$ as a matrix with entries $\phi(z)_{ij}$, then we require that $\phi(z)_{ij} \in \mathbb{R}[a, b]$ for all i, j. For a finite dimensional real vector space V, representation $\phi : \mathbb{C}^* \to GL(V)$ is called *algebraic* if after any isomorphism $V \cong \mathbb{R}^n$, the representation is algebraic.

The goal of this section is to show the following:

Proposition 2.1 (Hodge structures as representations)**.** *There is a bijection between the set of real Hodge structures of weight k and algebraic representations $\phi : \mathbb{C}^* \mapsto GL(V)$ such that $\phi(t)$ acts as $t^k \cdot v$ for $t \in \mathbb{R}^*$.*

We break the proof into several steps.

Lemma 2.2 (Representation from a Hodge structure)**.** *Given a Hodge structure V the representation $\phi : \mathbb{C}^* \to GL(V_{\mathbb{C}})$ given by $\phi(z)(v) = z^p \overline{z}^q \cdot v$ for $v \in V^{p,q}$, and extending linearly, induces, via restricting to $V \subseteq V_{\mathbb{C}}$, a real algebraic representation $\phi : \mathbb{C}^* \to GL(V)$. If V is of weight k, then $\phi(t) = t^k$ for all $t \in \mathbb{R}^*$.*

Proof. Given any Hodge structure V we can define a real representation $\phi : \mathbb{C}^* \to GL(V)$ of the multiplicative group of nonzero complex numbers as follows. Let $\phi(z)$ by the map defined by $\phi(z)(v) = z^p \overline{z}^q \cdot v$ for $v \in V^{p,q}$ extended linearly. To see that this representation is real consider a real vector $v \in V_{\mathbb{C}}$ then $v = \sum_{i,j} v_{i,j}$ for some $v_{i,j} \in V^{i,j}$. Since the sum is real the $v_{i,j}$ come in conjugate pairs, i.e. $\overline{v_{i,j}} = v_{j,i}$. But after acting by $\phi(z)$ the vector $\phi(z) \cdot v$ is still a sum of conjugate pairs since $\overline{z^p \overline{z}^q \cdot v_{i,j}} = \overline{z^p \overline{z}^q} \cdot \overline{v_{i,j}} = z^q \overline{z}^p \cdot \overline{v_{i,j}} = z^q \overline{z}^p \cdot v_{j,i}$. This shows that $\phi(z)(v)$ is real and so restricting ϕ to $V \subset V_{\mathbb{C}}$ we get a real representation.

Clearly the map $z \mapsto z^p \bar{z}^q$ is algebraic in a and b. One can then deduce that ϕ is algebraic.

If V is weight k then note that under this representation $\phi(t)$ acts as multiplication by t^k if $t \in \mathbb{R}^*$. □

Lemma 2.3 (Hodge structure from a representation)**.** *Let $\phi : \mathbb{C}^* \to GL(V)$ be a real algebraic representation. Letting $\phi_{\mathbb{C}} : \mathbb{C}^* \to GL(V_{\mathbb{C}})$ be the induced representation, then $V_{\mathbb{C}} = \bigoplus_{p+q} H^{p,q}$, where*

$$H^{p,q} = \{v \in V_{\mathbb{C}} : \; \phi(z)(v) = z^p \bar{z}^q \cdot v \quad \forall z \in \mathbb{C}^*\}$$

is the p, q-weight space for ϕ. This gives V the structure of a real Hodge structure. Moreover, if $\phi(t) = t^k$ for all $t \in \mathbb{R}^*$, then V is of weight k.

The key point is the following lemma, whose proof we temporarily postpone.

Lemma 2.4. *Any continuous homomorphism $\lambda : \mathbb{C}^* \to \mathbb{C}^*$ is given by $\lambda(z) = z^p \bar{z}^q$ for some $p, q \in \mathbb{Z}$.*

Proof of Lemma 2.3. The representation ϕ induces a representation $\phi_{\mathbb{C}}$: $\mathbb{C}^* \to GL(V_{\mathbb{C}})$. As $\mathbb{C}^*$ is abelian, this representation decomposes into a direct sum of 1-dimensional representations, which by virtue of Lemma 2.4 are of the form $z \mapsto z^p \overline{z}^q$ where $p + q$. Setting $H^{p,q} = \{v \in V_{\mathbb{C}} : \phi(z)(v) = z^p \overline{z}^q \cdot v \; \forall z \in \mathbb{C}^*\}$ to be the direct sum of the corresponding 1-dimensional representations, we have $V_{\mathbb{C}} = \bigoplus_{p+q} H^{p,q}$. Furthermore, by definition of the conjugate vector space, we have that $\phi(z)$ acts on $\overline{H^{p,q}}$ as $\overline{z^p\overline{z}^q} = z^q\overline{z^p}$. This shows that $\overline{H^{p,q}} = H^{q,p}$ hence $V_{\mathbb{C}} = \bigoplus_{p+q} H^{p,q}$ is a Hodge decomposition. Note that restricting to $\mathbb{R}^* \subseteq \mathbb{C}^*$ we see that if $\phi(t) = t^k$ for all $t \in \mathbb{R}^*$, then $p + q = k$. ☐

We can now give the proof of Proposition 2.1:

Proof of Proposition 2.1. One checks that the constructions in Lemma 2.2 and Lemma 2.3 are inverses of one another. ☐

2.1. **Proof of Lemma 2.4.** If $\lambda : \mathbb{C}^* \mapsto \mathbb{C}^*$ is an algebraic representation then $\lambda(x+iy)$, can be written as $p_1(x,y)+ip_2(x,y)$ with p_1, p_2 polynomials in x and y. But we have $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$ which means that $\lambda(z)$ is a polynomial in z, and $\overline{z}$. Note that any endomorphism of $\mathbb{C}^*$ of the form $z \mapsto z^m$ for m an integer is continuous. The conjugate map $z \mapsto \overline{z}$ is also a continuous endomorphism of $\mathbb{C}^*$. This shows that any endomorphism of $\mathbb{C}^*$ s.t. $\lambda(z)$ is a polynomial in z and $\overline{z}$ is continuous. In particular we can assume that our algebraic representation is a continuous endomorphism of $\mathbb{C}^*$. We will now show that any continuous endomorphism of $\mathbb{C}^*$ maps S^1 to itself. First suppose that $z \in S^1$ is an n-th root of unity for some integer $n \geq 0$, then we have $1 = \lambda(1) = \lambda(z^n) = \lambda(z)^n$ so that $\lambda(z)$ is an n-th root of unity. This shows that every root of unity is mapped into S^1. Note that the roots of unity have the form $e^{2\pi i\frac{p}{q}}$ for all rational numbers $\frac{p}{q}$. Hence for any $e^{2\pi i\frac{p}{q}}$ its image under λ lies in S^1. But the rational numbers are dense in the interval $[0,1]$ so there is a sequence of points $e^{2\pi i\frac{p_j}{q_j}}$ for $j \geq 1$ which converges to $e^{2\pi i a}$ for any $a \in [0,1]$. Then from the above $\lambda(e^{2\pi i\frac{p_j}{q_j}}) \in S^1$ for all j so that by the continuity of S^1 we must have $\lambda(e^{2\pi a}) \in S^1$. This shows that in order to classify the 1-dimensional algebraic representations of $\mathbb{C}^*$ we can consider the continuous homomorphisms from S^1 to S^1. We will utilize the work of Artin in which he uses two lemmas to show that the continuous homomorphisms $S^1 \to S^1$ all have the form $e^{it} \mapsto e^{iat}$ for some integer $a \in \mathbb{Z}$.

Lemma 2.5. *Consider $\mathbb{R}$ as a group under addition. Then the continuous homomorphisms $\phi : \mathbb{R} \mapsto \mathbb{R}$ are all of the form $\phi(x) = cx$ for some $c \in \mathbb{R}$.*

Proof. If ϕ is as above then for any integer $n \in \mathbb{Z}$ we have $\phi(n) = \phi(1 + 1 + ... + 1) = \phi(1) + \phi(1) + ... + \phi(1) = n\phi(1)$. Now let $\frac{n}{m}$ be any rational number and note that

$$m\phi(\frac{n}{m}) = \phi(\frac{mn}{m}) = \phi(n) = n\phi(1)$$

so that dividing both sides by m gives $\phi(\frac{n}{m}) = \frac{n}{m}\phi(1)$. Hence for all rational numbers q we have $\phi(q) = q\phi(1)$. But for any $x \in \mathbb{R}$ there is a sequence of rational numbers converging to $\mathbb{R}$ since the rational numbers are a dense subset of $\mathbb{R}$. Thus by the continuity of ϕ we must have $\phi(x) = x\phi(1)$ so that setting $c = \phi(1)$ we see that ϕ has the form $\phi(x) = cx$.

Lemma 2.6. *Consider $\mathbb{R}$ as a group under addition and S^1 as a group under multiplication. Then the continuous homomorphisms $\psi : \mathbb{R} \mapsto S^1$ all have the form $\phi(x) = e^{icx}$*

Corollary 2.7. *If $\lambda : S^1 \mapsto S^1$ is a continuous homomorphism then $\lambda(e^{ix}) = e^{inx}$ for some $n \in \mathbb{Z}$*

If λ is as above then $\lambda \circ exp$ is a continuous homomorphism from $\mathbb{R}$ to S^1. Therefore from the corollary we have $\lambda(e^{ix}) = e^{icx}$. Furthermore since λ is a homomorphism it sends the multiplicative identity of S^1 to itself so that $\lambda(e^{2\pi i}) = \lambda(1) = 1$. On the other hand $\lambda(e^{2\pi i}) = e^{2\pi ic}$ which implies that $e^{2\pi c} = 1$ and this happens iff $c = n$ for some $n \in \mathbb{Z}$.

Now we return to the problem of classifying the irreducible algebraic representations $\lambda : \mathbb{C}^* \mapsto \mathbb{C}^*$. We have seen that any such λ is a polynomial in z and $\overline{z}$, i.e. $\lambda(z) = a_1 z^{p_1}\overline{z}^{q_1} + ... + a_d z^{p_d}\overline{z}^{q_d}$. We have seen that restricting λ to S^1 defines a continuous homomorphism from S^1 to itself. On the one hand $\lambda(e^{it}) = a_1 e^{i(p_1-q_1)t} + ... + a_k e^{i(p_d-q_d)t}$, but on the other hand we have shown that $\lambda(e^{it}) = e^{int}$ which implies that $a_1 e^{i(p_1-q_1)t} + ... a_d e^{i(p_d-q_d)t} = e^{int}$. Note that in this context we are viewing the terms be^{irt} as functions defined by $e^{it} \mapsto be^{irt}$. However from Artin the irreducible characters of S^1, i.e. the n-th power maps e^{int} form a basis for the vector space of functions continuous functions $S_1 \mapsto \mathbb{C}$. Hence the equality $a_1 e^{i(p_1-q_1)t} + ... + a_k e^{i(p_k-q_k)t} = e^{int}$ can only be true of there exists j such that $p_j - q_j = k$, $a_j = 1$, and $a_i = 0$ whenever $i \neq j$. This is because otherwise we would have a nontrivial linear dependence between the functions e^{irt} for r an integer. In other words we must have $\lambda(z) = z^p\overline{z}^q$ for some $p, q \in \mathbb{Z}$.

3. Hodge structures via Filtrations

Definition 3.1. Let V be an $\mathbb{R}$-vector space. Then a sequence

$$V_{\mathbb{C}} ... \supset F^i(V) \supset F^{i+1}(V) \supset ...$$

is called a filtration of $V_{\mathbb{C}}$

Proposition 3.2. *There is a bijection between Hodge structures of weight k and filtrations of $V_{\mathbb{C}}$ such that $\cup_{i\in\mathbb{Z}} F^i(V) = V_{\mathbb{C}}$ and $F^i(V) \cap F^j(V) = 0$ if $i + j = k + 1$.*

Given a Hodge decomposition $V_{\mathbb{C}} = \oplus_{i+j=k} V^{i,j}$ let

$$F^k(V) = \bigoplus_{i\geq k} V^{i,j}$$

Clearly by definition we have $V_{\mathbb{C}} ... \supset F^k(V) \supset F^{k+1}(V) \supset ...$ Furthermore suppose $p + q = k + 1$ then we have

$$F^p(V) \cap \overline{F^q(V)} = \bigoplus_{i\geq p} V^{i,j} \cap \overline{\bigoplus_{r\geq q} V^{r,s}} = \bigoplus_{i\geq p\ r\geq q} V^{i,j} \cap V^{s,r}$$

Given arbitrary i and r in the above equation we have $i \geq p$ and $r \geq q$ so that

$$i + r \geq p + q = k + 1 > s + r = k$$

but then $i \neq s$ so that $V^{i,j} \cap V^{s,r} = 0$ which from the above implies $F^p(V) \cap \overline{F^q(V)} = 0$. Clearly $\cup_{i\in\mathbb{Z}} F^i(V) = V_{\mathbb{C}}$.

On the other hand suppose we have a filtration $F^i(V)$ such that $\cup_{i\in\mathbb{Z}} F^i(V) = V_{\mathbb{C}}$ and $F^i(V) \cap F^j(V) = 0$ if $i + j = k + 1$. Set $V^{i.j} = F^i(V) \cap \overline{F^j(V)}$, then we must show that $V_{\mathbb{C}} = \bigoplus_{i+j=k} V^{i,j}$. First suppose that $i \neq r$ then we have

$$V^{i,j} \cap V^{r,s} = F^i \cap \overline{F^j(V)} \cap F^r(V) \cap \overline{F^s(V)}$$

WLOG assume $i > r$ so that $j < s$. Then we must have $i+s > i+j = k$ so that

$$V^{i,j} \cap V^{r,s} = F^i(V) \cap \overline{F^s(V)} \subseteq F^i(V) \cap \overline{F^{k-i+1}(V)} = 0$$

This shows that $\bigoplus_{i,j \in \mathbb{Z}} V^{i,j} \leq V_{\mathbb{C}}$. Let $v \in V_{\mathbb{C}}$ then because $\cup_{i \in \mathbb{Z}} F^i(V) = V_{\mathbb{C}}$ we have $v \in F^i(V)$ for some i. Since $F^i(V) = \bigoplus_{p \geq i} V^{p,q}$ it follows that $v \in \bigoplus_{i,j \in \mathbb{Z}} V^{i,j}$. To finish the proposition we must show that the decomposition has weight k. This follows from the condition $F^p \cap \overline{F^q} = 0$ when $p+q = k+1$ because if $i+j > k$ then

$$V^{i,j} = F^i \cap \overline{F^j} \subseteq F^i \cap \overline{F^{k-i+1}} = 0$$

This completes the proof.

So far we have seen three equivalent definitions of a Hodge structure. We have also seen how to go back and forth from the Hodge decomposition to an algebraic representation or a filtration of $V_{\mathbb{C}}$. To go between filtrations and representations we can use their shared Hodge decomposition. Then if $\rho : \mathbb{C}^* \mapsto GL(V)$ is an algebraic representation, with $\rho(t)$ acting as multiplication by t^k, then setting $F^i(V) = \bigoplus_{p \geq i} V^{p.q}$ where $V^{p.q} = \{v \in V_{\mathbb{C}} \mid \rho(v) = z^p \overline{z}^q \cdot v\}$ gives a filtration of V_C. Using the induced Hodge decomposition one can check that this filtration has the properties required by the previous proposition. On the other hand, given a filtration of $V_{\mathbb{C}}$ such that $\cup_{i \in \mathbb{Z}} F^i(V) = V_{\mathbb{C}}$ and $F^i(V) \cap F^j(V) = 0$ if $i+j = k+1$. We define a representation of C^* on $V_{\mathbb{C}}$ as $\rho_{\mathbb{C}}(v) = z^p \overline{z^p} \cdot v$ for $v \in F^p(V) \cap \overline{F^q(V)}$ and extend this map linearly using the decomposition of proposition. This representation is clearly algebraic and an argument similar to the one before proposition J shows that $\rho_{\mathbb{C}}$ restricts to a real algebraic representation of C^* on $V_{\mathbb{R}}$ where $\rho(t)$ acts as t^k.

4. Almost Complex Structures

Definition 4.1. Let V be a real vector space and let $id : V \mapsto V$ be the identity map on V. An almost complex structure on V is an endomorphism $I : V \mapsto V$ such that $I^2 = -id$.

Any $I : V \mapsto V$ almost complex structure on a real vector space $(V, \cdot)$ defines a complex vector space $(V, *)$ under the action

$$(a + bi) * v = a \cdot v + b \cdot I(v)$$

for all $v \in V$. This follows from the fact that V is an abelian group, and the following module conditions

(1)

$$(a'+b'i)(a+bi)*v = (a'a-b'b)+(a'b+ab')i \cdot v = (a'a-b'b) \cdot v+(a'b+ab') \cdot I(v) =$$

$$(a' + b'i) * (a \cdot v + b \cdot I(v)) = (a' + b'i) * ((a + bi) * v)$$

(2)

$$(a + bi + a' + b'i) * v = (a + a') \cdot v + (b + b') \cdot I(v) =$$

$$(a \cdot v + b \cdot I(v)) + (a' \cdot v + b' \cdot I(v)) = (a + bi) * v + (a' + b'i) * v$$

(3)

$$(a + bi) * (v + v') = a \cdot (v + v') + b \cdot I(v + v') =$$

$$a \cdot v + b \cdot I(v) + a \cdot v' + b \cdot I(v') = (a + bi) * v + (a' + b'i) * v'$$

Conversely suppose $(V, *)$ is a complex vector with $I : V \mapsto V$ defined by $I(v) = i * v$ for all $v \in V$. Then because $I^2(v) = i^2 * v = (-1) * v = -v$ it follows that I is an almost complex structure on the real vector space underlying $(V, *)$. This shows that almost complex structures and complex vector spaces are equivalent notions.

Proposition 4.2. *If V is a finite dimensional real vector space for which there exists an almost complex structure $I : V \mapsto V$ then the dimension of V is even.*

Proof. By the above any almost complex vector space V has a complex vector space structure. Since V is finite dimensional we have $V \cong \mathbb{C}^n$ for some n. But $\mathbb{C} \cong \mathbb{R}^2$ as a real vector space which implies that $V \cong \mathbb{R}^{2n}$. $\square$

Definition 4.3. For a real vector space V we define $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Note that V can be considered a real subspace of $V_{\mathbb{C}}$ via the map $v \mapsto v \otimes 1$. Furthermore if I is an almost complex structure on V then we can also denote by I the $\mathbb{C}$-linear map on $V_{\mathbb{C}}$ defined by $v \otimes z \mapsto I(v) \otimes z$.

Proposition 4.4. *The only eigenvalues of I on $V_{\mathbb{C}}$ are i and $-i$.*

Proof. The map I on $V_{\mathbb{C}}$ satisfies $I^2 + id = 0$ so its minimal polynomial divides $x^2 + 1$. This shows that the only possible eigenvalues for I are i and $-i$. Furthermore for any $v \in V$ we have that $\frac{1}{2}(v - iI(v))$ and $\frac{1}{2}(v + iI(v))$ are eigenvectors for I with eigenvalues i and $-i$ respectively.

Definition 4.5. If I is the $\mathbb{C}$-linear extension to $V_{\mathbb{C}}$ of an almost complex structure on V then $V^{1,0}$ and $V^{0,1}$ denote the i and $-i$ eigenspaces respectively.

$$V^{1,0} = \{v \in V_{\mathbb{C}} \mid I(v) = i \cdot v\} \qquad V^{0,1} = \{v \in V_{\mathbb{C}} \mid I(v) = -i \cdot v\}$$

Lemma 4.6.

$$V^{1,0} = \{\frac{1}{2}(v - iI(v)) \mid v \in V\}$$

$$V^{0,1} = \{\frac{1}{2}(v + iI(v) \mid v \in V\}$$

Proof. Given any $v \in V_{\mathbb{C}}$ we have

$$v = \sum_j v_j \otimes (a_j + ib_j) = \sum_j v_j \otimes a_j + \sum_j v_j \otimes ib_j = (\sum_j a_j v_j) \otimes 1 + (\sum_j b_j v_j) \otimes i$$

where the last equality follows because we are tensoring over $\mathbb{R}$ so that $v \otimes r = rv \otimes 1$ for all $r \in \mathbb{R}$. This shows that elements of $V_{\mathbb{C}}$ can be written as $x + iy$ for $x, y \in V$. Note that we are using the shorthand notation $x = x \otimes 1$ and $iy = y \otimes i$, i.e. identifying V with the real subspace $V \otimes 1 = \{v \otimes 1 : v \in V\}$ in $V_{\mathbb{C}}$. Then for the first assertion if $v \in V^{1,0}$ we must have $I(v) = I(x + iy) = I(x) + iI(y)$ On the other hand

$$I(v) = i(x + iy) = -y + ix$$

This shows that $y = -I(x)$ hence $v = \frac{1}{2}(2x - iI(2x))$ which proves that $V^{1,0} \subset \{\frac{1}{2}(v - iI(v)) \mid v \in V\}$. The fact that for any $v \in V$ we have

$$I(\frac{1}{2}(v - iI(v))) = \frac{1}{2}I(v - iI(v)) = \frac{1}{2}(iv + I(v)) = i \cdot \frac{1}{2}(v - iI(v))$$

proves the other inclusion. The proof of the second assertion is analogous.

Proposition 4.7. *Let V be a real vector space with an almost complex structure I. Then we have*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

Furthermore the complex conjugation map provides an isomorphism of $V^{1,0}$ and $V^{0,1}$ as $\mathbb{R}$-vector spaces.

Proof. It is clear that $V^{1,0} \cap V^{0,1} = \{0\}$. Let $v_1, ..., v_d$ be an $\mathbb{R}$ basis for V and hence a $\mathbb{C}$ basis for $V_{\mathbb{C}}$. Then for each v_j we have $\frac{1}{2}(v_j - iI(v_j)) \in V^{1,0}$ and $\frac{1}{2}(v_j + iI(v_j)) \in V^{0,1}$. Therefore it follows that

$$v_j = \frac{1}{2}(v_j - iI(v_j)) + \frac{1}{2}(v_j + iI(v_j)) \in V^{1,0} \oplus V^{0,1}$$

But any element of $v \in \mathbb{C}$ is a $\mathbb{C}$-linear combination of the basis, i.e.

$$v = a_1 v_1 + ... + a_d v_d =$$

$$\frac{1}{2}a_1(v_1 - iI(v_1)) + \frac{1}{2}a_1(v_1 + iI(v_1)) + ... + \frac{1}{2}a_d(v_d - iI(v_d)) + \frac{1}{2}a_d(v_d + iI(v_d))$$

which shows that $v \in V^{1,0} \oplus V^{0,1}$ so that $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Since any $v \in V_{\mathbb{C}}$ can be written as $v = x+iy$ for $x, y \in V$ the complex conjugation map acts on $V_{\mathbb{C}}$ as $\overline{x+iy} = x - iy$. Therefore given any $v \in V$ we have $\overline{\frac{1}{2}(v - iI(v))} = \frac{1}{2}(v + iI(v)) \in V^{0,1}$. Using this and the lemma shows that the complex conjugation map interchanges the two eigenspaces. This interchanging gives an $\mathbb{R}$-linear map between the eigenspaces since for any $\frac{1}{2}(v_1 - iI(v_1)), \frac{1}{2}(v_2 - iI(v_2)) \in V^{1,0}$ and $r \in \mathbb{R}$ we have

$$\overline{\frac{1}{2}(v_1 - iI(v_1)) + r \cdot \frac{1}{2}(v_2 - iI(v_2))} = \overline{\frac{1}{2}(v_1 + r \cdot v_2) - \frac{1}{2}i(I(v_1) + r \cdot I(v_2))} =$$

$$\frac{1}{2}(v_1 + r \cdot v_2) + \frac{1}{2}i(I(v_1) + r \cdot I(v_2)) = \frac{1}{2}(v_1 + iI(v_1)) + r \cdot \frac{1}{2}(v_2 + iI(v_2))$$
$$= \overline{\frac{1}{2}(v_1 + iI(v_1))} + r \cdot \overline{\frac{1}{2}(v_2 + iI(v_2))}$$

Furthermore, conjugation from $V^{1,0}$ to $V^{0,1}$ must in fact be an isomorphism of $\mathbb{R}$-vector spaces since it has a two-sided inverse given by conjugation from $V^{0,1}$ to $V^{1,0}$.

$\square$

Recall the relationship between algebraic representations and Hodge structures of weight k given by proposition 2.1. Given a Hodge structure of weight 1, i.e. a decomposition $V_{\mathbb{C}} = V^{1,0} \bigoplus V^{0,1}$ for a real vector space $V_{\mathbb{R}}$, we obtain an algebraic representation $\phi : \mathbb{C}^* \to GL(V_{\mathbb{R}})$ as above. But we must have $\phi(-1) = -Id$ since -1 is supposed to act as multiplication by -1. Therefore $J = \phi(i)$ has the property that $J^2 = \phi(i)\phi(i) = \phi(i^2) = \phi(-1) = -Id$ so that J is an almost complex structure. Moreover, if we have an almost complex structure J, then we have already seen that J determines a decomposition $V_{\mathbb{C}} = V^{1,0} \bigoplus V^{0,1}$ where the summands are the i and $-i$ eigenspaces respectively. Note that we obtained an algebraic representation ϕ from a Hodge structure of weight k by specifying that $z \in \mathbb{C}^*$ act on $V^{p,q}$ by $z^p \overline{z}^q$ and then restricting this to an action on V. But this implies that in the decomposition above the extension of $\phi(i)$ to $V_{\mathbb{C}}$ acts on $V^{1,0}$ and $V^{0,1}$ as multiplication by i and $-i$ respectively. Note that any $v \in V$ is a sum of elements in $V^{1,0}$ and $V^{0,1}$. Then $J = \phi(i)$ because these operators act the same on both $V^{1,0}$ and $V^{0,1}$. This shows that the almost complex structure obtained via the algebraic representation is the same as the J which determined the Hodge structure in the first place. In particular we have a bijective correspondence between almost complex structures on V and Hodge structures of weight 1 on V.

Let V be an $\mathbb{R}$-vector space with an almost complex structure I. Then there is an almost complex structure on the dual space V^* given by $I(f)(v) = f(I(v))$. To see this note that the map I on V^* is linear since for any $f_1, f_2 \in V^*$ and $r \in \mathbb{R}$ we have

$$I(f_1+r\cdot f_2)(v) = (f_1+r\cdot f_2)(I(v)) = f_1(I(v))+r\cdot f_2(I(v)) = I(f_1)(v)+r\cdot I(f_2)(v)$$

Furthermore for any $v \in V$ and $f \in V^*$

$$I^2(f)(v) = f(I^2(v)) = f(-v) = -f(v)$$

shows that I is an almost complex structure on V^*.

From proposition 4.7 we must have that $(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$ where

$$(V^*)^{1,0} = \{f \in V^* \mid f(I(v)) = i \cdot f(v) \; \forall \; v \in V\}$$

$$(V^*)^{0,1} = \{f \in V^* \mid f(I(v)) = -i \cdot f(v) \; \forall \; v \in V\}$$

It should also be noted that $(V^*)_{\mathbb{C}} \cong (V_{\mathbb{C}})^*$ via the map Φ which sends $f \otimes z_0$ to the map $f_{\mathbb{C}} \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ defined by $v \otimes z \mapsto f(v) \otimes z_0 z$. Note that here we are using the identification $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}$ as real vector spaces. To see the map Φ is an isomorphism let V be a finite dimensional real vector space and choose a basis $v_1, ..., v_d$. Then $v_1 \otimes 1, ..., v_d \otimes 1$ is a $\mathbb{C}$-basis of $V_{\mathbb{C}}$. Also if we let v^i denote elements of the dual basis then $v^1 \otimes 1, ..., v^d \otimes 1$ is a $\mathbb{C}$-basis of $(V^*)_{\mathbb{C}}$. But then $\Phi(v^i \otimes 1)(v_j \otimes 1) = v^i(v_j) \otimes 1 = \sigma_{ij} \otimes 1 = \sigma_{ij}$ so that $\Phi(v^i \otimes 1) = (v_i \otimes 1)^{\vee}$. This implies that Φ maps a $\mathbb{C}$-basis of $(V^*)_{\mathbb{C}}$ to a $\mathbb{C}$-basis of $(V_{\mathbb{C}})^*$ so that Φ is an isomorphism. Another identification that is worth remarking on is $(V^*)^{1,0} = \mathrm{Hom}_{\mathbb{C}}((V, I), \mathbb{C})$. This follows easily from the definition of $(V^*)^{1,0}$ since its elements are exactly the maps from (V, I) to $\mathbb{C}$ which preserve the complex vector space structure. In the same way we can identify $(V^*)^{0,1}$ with the $\overline{\mathbb{C}}$-linear maps from (V, I) to $\mathbb{C}$ which are the same as $\mathbb{C}$-linear maps from (V, I) to $\overline{\mathbb{C}}$.

Recall that the tensor product of K-vector spaces is defined so that it is universal with respect to multilinear maps. That is the tensor product is a vector space $\otimes^k V$ equipped with a map $\phi : V^k \mapsto \otimes^k V$ such that every multilinear map $p : V^k \mapsto P$ factors uniquely through ϕ. In other words $p = f \circ \phi$ for some unique K-linear map $f : \otimes_k V \mapsto P$. Therefore according to Aluffi the tensor product can be thought of as the best approximation to V^k if we wish to view K-multilinear maps as K-linear. Often multilinear maps have additional properties one example being alternating maps. If $p : V^k \mapsto P$ is a multilinear map then p is said to be alternating if $\phi(x_1, x_2, ..., x_k) = 0$ whenever $x_i = x_j$ for some $i \neq j$. For example, let I be the counterclockwise rotation by $\frac{\pi}{2}$ on $\mathbb{R}^2$. Then we have $x \cdot y = I(x) \cdot I(y)$ for all $x, y \in \mathbb{R}^2$ where $\cdot$ is the dot product. Furthermore we can define a map $\omega : \mathbb{R}^2 \mapsto \mathbb{R}$ by

$\omega(x, y) = x \cdot I(y)$. Note that for all $x \in \mathbb{R}^2$ we have

$$\omega(x, x) = x \cdot I(x) = I(x) \cdot I^2(x) = -(x \cdot I(x)) = -\omega(x, x)$$

This implies that $\omega(x, x) = 0$ for all x hence f is alternating. The existence of alternating multilinear maps motivates the following.

Definition 4.8. Let K be a field and V a K-vector space. Then the k-th exterior power $\bigwedge^k V$ is defined to be universal with respect to alternating multilinear maps from V^k. In other words there is a map $\phi : V^k \mapsto \bigwedge^k V$ such that give any alternating map $p : V^k \mapsto P$ there is a unique K-linear map $f : \bigwedge^k V \mapsto P$ with $p = f \circ \phi$.

The tensor product can be constructed starting with the free K-module V^k. Then $\otimes^k V$ is the quotient of the free module by the submodule generated by the elements

$$(x_1, ..., x_{i_1} + x + i_2, ..., x_k) - (x_1, ..., x_{i_1}, ..., x_k) - (x_1, ..., x_{i_2}, ..., x_k)$$

$$(x_1, ..., r_i x_i, ..., x_k) - r_i(x_1, ..., x_i, ..., x_k)$$

for any $r_i \in K$ any $(x_1, ..., x_k) \in V^k$ and any $i \in \{1, ..., k\}$. The k-th exterior power is constructed by adding the elements $(x_1, ..., x_k)$ such that $x_i = x_j$ for some $i \neq j$ to the generating set. These constructions are essentially only useful in order to verify that such universal objects exist. When considering the tensor or exterior powers it is better to think of them as formal sums of elements in V^k with some special properties. For example, one important property of the tensor product is that $v \otimes 0 = 0$. This can be proved by noting that $v \otimes 0 = v \otimes (0 + 0) = v \otimes 0 + v \otimes 0$. Since exterior powers can be constructed as the quotient of the corresponding tensor power this property also holds in $\bigwedge^k V$. Another important property of exterior powers has to do with permutations.

Lemma 4.9. *If V is a K-vector space then we have*

$$x_1 \wedge ... \wedge x_i \wedge x_j \wedge ... \wedge x_k = -x_1 \wedge ... \wedge x_j \wedge x_i \wedge ... \wedge x_k$$

for all $x_1 \wedge ... \wedge x_k \in \bigwedge^k V$ and all $i < j$.

Proof. By definition of the exterior power we have

$$x_1 \wedge ... \wedge (x_i + x_j) \wedge (x_i + x_j) \wedge ... \wedge x_k = 0$$

on the other hand using multi linearity and that $x \wedge x = 0$ gives

$$x_1 \wedge ... \wedge (x_i + x_j) \wedge (x_i + x_j) \wedge ... \wedge x_k =$$

$$(x_1 \wedge ... \wedge x_i \wedge x_j \wedge ... \wedge x_k) + (x_1 \wedge ... \wedge x_j \wedge x_i \wedge ... \wedge x_k) = 0$$

□

The above shows that we can interchange elements of an exterior power at the expense of changing the sign. Therefore suppose $x_1 \wedge ... \wedge ... x_k \in \bigwedge^k V$ and $\sigma \in S_k$. Then $x_{\sigma(1)} \wedge ... \wedge x_{\sigma(k)}$ is obtained by interchanging elements.This can be done because every element of S_k is a product of transpositions. Furthermore, this combined with the previous lemma prove the following.

Proposition 4.10. *Given any $\sigma \in S_k$ we have*

$$x_1 \wedge ... \wedge x_k = sgn(\sigma) x_{\sigma(1)} \wedge ... \wedge x_{\sigma(k)}$$

for all $x_1 \wedge ... \wedge x_k \in \bigwedge^k V$

The k-th exterior powers can all be considered as subspaces of a larger structure. This structure is called the exterior algebra.

Definition 4.11. If V is a K-vector space then we define the exterior algebra by

$$\wedge^* V = K \oplus V \oplus \wedge^2 V \oplus ...$$

where the multiplication operation of the algebra is $\wedge$.

Proposition 4.12. *if V is a d dimensional K-vector space then*

$$\bigwedge^{*} V = \bigoplus_{k=0}^{d} \bigwedge^{k} V$$

Proof. If $x_1 \wedge ... \wedge x_n \in \bigwedge^n V$ for $n > d$ then the set of vectors $\{x_1, ..., x_n\}$ is linearly dependent. Therefore we must have $x_n = a_1 x_1 + ... a_{n-1} x_{n-1}$ for some coefficients in K. It follows that

$$x_1 \wedge ... \wedge x_{n-1} \wedge x_n = x_1 \wedge ... \wedge x_{n-1} \wedge (a_1 x_1 + ... + a_{n-1} x_{n-1}) =$$

$$\sum_{k=1}^{n-1} a_k (x_1 \wedge ... \wedge x_{n-1} \wedge x_k) = 0$$

This shows that $\bigwedge^n V = 0$ for $n > d$ and this together with the definition of the exterior algebra proves the proposition. $\square$

If V is real of dimension d then $V_{\mathbb{C}}$ has dimension d as a $\mathbb{C}$-vector space. Then proposition 4.12 says that

$$\bigwedge^{*} V_{\mathbb{C}} = \bigoplus_{k=0}^{d} \bigwedge^{k} V_{\mathbb{C}}$$

Note that $\bigwedge^* V_{\mathbb{C}}$ is isomorphic to $(\bigwedge^* V)_{\mathbb{C}}$ via the map defined by

$$(x_1 \otimes z_1) \wedge ... \wedge (x_k \otimes z_k) \mapsto (x_1 \wedge ... \wedge x_k) \otimes (z_1 ... z_k)$$

This means that $\bigwedge^* V$ is a real subspace of $\bigwedge^* V_{\mathbb{C}}$ and in fact it is the subspace left invariant under the complex conjugation map $(x \otimes z) \mapsto (x \otimes \bar{z})$. Furthermore, the above isomorphism shows that the complex conjugation map is multiplicative, in other words $\overline{x \wedge y} = \overline{x} \wedge \overline{y}$. If V has a complex structure then we can use proposition 4.7 to further decompose the exterior algebra of $V_{\mathbb{C}}$.

Definition 4.13. If I is an almost complex structure on V with $V^{1,0}$ and $V^{0,1}$ as in proposition 4.7 then we define

$$\bigwedge^{p,q} V = \bigwedge^{p} V \otimes_{\mathbb{C}} \bigwedge^{q} V$$

where the exterior products are taken over $\mathbb{C}$

Let $\{e_1, ..., e_d\}$ be a basis of V. Then the elements $e_{i_1} \wedge ... \wedge e_{i_k}$ span $\bigwedge^k V$ for $i_1, ..., i_k \in \{1, ...d\}$. By the alternating condition each of these elements is a multiple of some $e_{j_1} \wedge ... \wedge e_{j_k}$ where $j_1 < ... < j_k$. Therefore the latter elements must also be a spanning set for $\bigwedge^k V$, we denote this set B. One proof that B is linearly independent uses the construction of the exterior algebra from the free K-module $F(V^k)$. In particular one must show that no linear combination $\sum_J a_J e_J$ for $J \subset \{1, ..., d\}$ can also be a linear combinations of elements in the generating set mentioned before lemma 0.6. The details are tedious to write out fully but not too difficult. Furthermore, note that any vector in B is uniquely determined by a k-element subset of $\{1, ..., d\}$. This shows that $|B| = \binom{d}{k}$ proving the following result

Lemma 4.14. *If V has dimension d over a field K then $\bigwedge^k V$ has dimension $\binom{d}{k}$ where $k < d$*

Proposition 4.15. *Let V be a real vector space with an almost complex structure.*

(1) *$\bigwedge^{p,q} V$ is isomorphic to a subspace of $\bigwedge^{p+q} V_{\mathbb{C}}$*

(2) $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$

(3) *Complex conjugation provides a $\mathbb{C}$-antilinear isomorphism between $\bigwedge^{p,q} V$ and $\bigwedge^{q,p} V$. Therefore we have $\overline{\bigwedge^{p,q} V} = \bigwedge^{q,p} V$.*

Proof. (1) Let $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ be bases of $V^{1,0}$ and $V^{0,1}$ respectively. Consider the subspace V' of $\bigwedge^{p+q} V_{\mathbb{C}}$ generated by elements of the form $v_{i_1} \wedge ... \wedge v_{i_p} \wedge w_{j_1} \wedge ... \wedge w_{j_q}$ with $i_1 < ... < i_p$ and $j_1 < ... < j_q$. By an argument similar to the one used in the proof of lemma 0.9 it can be seen that these vectors are linearly independent. Furthermore there are $\binom{n}{p}\binom{n}{q}$ of them so $\binom{n}{p}\binom{n}{q}$ is the dimension of V'. Note that if V and W have dimensions n_1 and n_2 respectively then $V \otimes W$ has dimension $n_1 n_2$. This together with the lemma shows that $\bigwedge^{p,q} V$ also has dimension $\binom{n}{p}\binom{n}{q}$. Therefore $\bigwedge^{p,q} V$ is isomorphic to the subspace V' of $\bigwedge^{p+q} V_{\mathbb{C}}$.

(2) With notation as in part 1 we have that $\{v_1, ..., v_n, w_1, ..., w_n\}$ is a basis of $V_{\mathbb{C}}$. From the first part the elements of $\bigcup_{p+q=k} \bigwedge^{p,q} V_{\mathbb{C}}$ clearly generate $\bigwedge^k V_{\mathbb{C}}$. Hence we must show that $\bigwedge V^{p,q} \cap \bigwedge V^{r,s} = \{0\}$ when $r \neq s$. Suppose that $x_1 \wedge ... \wedge x_d \in \bigwedge V^{p,q} \cap \bigwedge V^{r,s}$ for $p \neq r$. WLOG suppose that $r < p$, then we have $x_{r+1} \in V^{1,0}$ and $x_{r+1} \in V^{0,1}$. However, $V^{1,0} \cap V^{0,1} = \{0\}$, which implies that $x_{r+1} = 0$. Hence it follows that

$$x_1 \wedge ... \wedge x_{r+1} \wedge ... \wedge x_n = x_1 \wedge ... \wedge 0 \wedge ... \wedge x_n = 0$$

This shows that $\bigwedge V^{p,q} \cap \bigwedge V^{r,s} = 0$ for $p \neq r$ completing the proof of part 2.

(3) Note that complex conjugation is multiplicative on exterior powers, in other words $\overline{x_1 \wedge x_2} = \overline{x_1} \wedge \overline{x_2}$. Furthermore from proposition C we have that $\overline{V^{1,0}} = V^{0,1}$.

Hence if $x_1 \wedge ... \wedge x_p \wedge y_1 \wedge ... \wedge y_q \in \bigwedge^{p,q} V$ then

$$\overline{x_1 \wedge ... \wedge x_p \wedge y_1 \wedge ... \wedge y_q} = \overline{x_1} \wedge ... \wedge \overline{x_p} \wedge \overline{y_1} \wedge ... \wedge \overline{y_q} \in \bigwedge^{q,p} V$$

Complex conjugation is clearly a bijection on $\bigwedge^* V_{\mathbb{C}}$ and it can be seen that this map is $\mathbb{C}$-antilinear. It follows that $\overline{\bigwedge^{p,q} V} = \bigwedge^{q,p} V$

□

Essentially the proposition proves that the subspaces $\bigwedge^{p,q} V$ make $\bigwedge^k V$ into a pure weight k Hodge structure. Note that in the case $k = 1$ we simply get back the decomposition of $V_{\mathbb{C}}$ given by proposition 4.7. This is a pure weight 1 Hodge structure on V.

Proposition 4.16. *Let V be an $\mathbb{R}$-vector space that is pure weight 1 Hodge structure with $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Then there exists an almost complex structure on V such that $V^{1,0}$ and $V^{0,1}$ are the i and $-i$-eigenspaces of I extended to $V_{\mathbb{C}}$*

Proof. For any $v \in V \subset V_{\mathbb{C}}$ we have $v = w_1 + w_2$ for some $w_1 \in V^{1,0}$ and some $w_2 \in V^{0,1}$. Then if $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$ we must have $x_1 + x_2 = v$ and $y_1 = -y_2$. Furthermore $x_2 - iy_2 \in V^{1,0}$ implies that $x_1 + x_2 + i(y_1 - y - 2) = v + i2y_1 \in V^{1,0}$. Hence we must also have $v - i2y_1 \in V^{0,1}$. But then $\frac{1}{2}(v + i2y_1)$ and $\frac{1}{2}(v - i2y_1)$ are the two unique vectors, in $V^{1,0}$ and $V^{0,1}$ respectively, whose sum is V. This shows that for every $v \in V$ there is a unique $y_1 = w \in V$ such that $v + iw \in V^{0,1}$. Then we can define a map $I : V \mapsto V$ by $I(v) = -w$. Note that since $V^{1,0}$ is closed under complex scaling we have $i(v + iw) = -w + iv$ so that $I^2(v) = I(-w) = -v$. Therefore I is an almost complex structure on V and we can consider its $\mathbb{C}$-linear extension to $V_{\mathbb{C}}$. If $x+iy \in V^{1,0}$ then $I(x) = -y$ and $I(y) = x$ so that $I(x + iy) = i(x) + iI(y) = -y + ix = i(x + iy)$. Also if $x+iy \in V_{\mathbb{C}}$ is an i-eigenvector then $I(x) = -y$ so that $x+iy \in V^{1,0}$ by definition of I. This shows that $V^{1,0}$ is the i-eigenspace of I and since $V^{0,1} = \overline{V^{1,0}}$ we also have that $V^{0,1}$ is the $-i$-eigenspace □

Lemma 4.17. *Let $z_i = x_i + iy_i$ for $i \in \{1, ..., d\}$ be a $\mathbb{C}$-basis for $V^{1,0}$. Then for any $m \leq d$ we have*

$$(-2i)^m (z_1 \wedge \overline{z_1}) \wedge ... \wedge (z_m \wedge \overline{z_m}) = (x_1 \wedge y_1) \wedge ... \wedge (x_m \wedge y_m)$$

For the dual basis z^i of $V^{1,0^*}$ we have

$$(\frac{i}{2})^m (z^1 \wedge \overline{z^1}) \wedge ... \wedge (z^m \wedge \overline{z_m}) = (x^1 \wedge y^1) \wedge ... \wedge (x^m \wedge y^m)$$

Definition 4.18. In view of proposition 4.15 we define the projections

$$\Pi^k : \bigwedge^* V_{\mathbb{C}} \mapsto \bigwedge^k V_{\mathbb{C}}$$

and

$$\Pi^{p,q} : \bigwedge^* V_{\mathbb{C}} \mapsto \bigwedge^k V_{\mathbb{C}}$$

Also the operator $\mathbf{I}$ will be defined on $\bigwedge^* V_{\mathbb{C}}$ by

$$\mathbf{I} = \sum_{p,q} i^{p-q} \Pi^{p,q}$$

In other words $\mathbf{I}$ is the endomorphism of $V_{\mathbb{C}}$ which acts like multiplication by i^{p-q} on the subspace $V^{p,q}$. The corresponding linear maps on the dual space are given the same notation.

Definition 4.19. Let $(V, \langle , \rangle)$ be an euclidean vector space. Then an almost complex structure I on V is called compatible if for all $v, w \in V$ we have $\langle v, w\rangle = \langle I(v), I(w)\rangle$ i.e. I is an orthogonal operator with respect to the scalar product on V.

Note that if I is a compatible almost complex structure then

$$\langle I(v), v\rangle = \langle I^2(v), I(v)\rangle = \langle -v, I(v)\rangle = -\langle I(v), v\rangle$$

this implies that $\langle I(v), v\rangle = 0$ so that v is always orthogonal to its image under I.

Suppose V is a real vector space of dimension two with fixed orientation. Then any scalar product $\langle , \rangle$ on V defines an almost complex structure as follows. Given any $v \in V$ the vector $I(v)$ is uniquely determined by the conditions $\langle v, I(v)\rangle = 0$, $|v| = |I(v)|$, and $\{v, I(v)\}$ has positive orientation. This is because the second condition determines the length of $I(v)$ and the first and third conditions determine its direction. In fact these conditions are equivalent to the definition of I as a rotation by $\frac{\pi}{2}$ which shows that I is indeed an almost complex structure. Furthermore it is not difficult to see that such an I must be compatible with the scalar product $\langle , \rangle$. Two scalar products $\langle , \rangle_1$ and $\langle , \rangle_2$ define the same almost complex structure on V if $\langle , \rangle_1 = \lambda \langle , \rangle_2$ for some $\lambda \in \mathbb{R}$. Indeed one can see that if $\langle v, I(v)\rangle_1$ and $|v|_1 = |I(v)|_1$ and $\{v, I(v)\}$ is positively oriented then these conditions are also satisfied on $(V, \langle , \rangle_2)$. The relationship among scalar products given by $\langle , \rangle_1 = \lambda \langle , \rangle_2$ for some $\lambda \in \mathbb{R}^*$ is an equivalence relation. This is because the existence of multiplicative inverses in $\mathbb{R}^*$ gives symmetry and the other axioms, reflexive and transitive, are easy consequences of the definition of the relationship. Two elements in the same equivalence class are called conformally equivalent and the above shows that there is a bijection between conformal equivalence classes and two dimensional almost complex structures on V.

Definition 4.20. Let $(V, \langle , \rangle)$ be a euclidean vector space with a compatible almost complex structure I. The the fundemental form of $(V, \langle , \rangle, I)$ is defined by

$$\omega(v, w) = -\langle v, I(w) \rangle = \langle I(v), w \rangle$$

Lemma 4.21. *If $(V, \langle , \rangle, I)$ and ω is the fundemental form then $\omega \in \bigwedge^2 V^* \cap \bigwedge^{1,1} V^*$*

Proof. ω is alternating since $\omega(v, w) = -\omega(w, v)$ by definition. Furthermore we have

$$(\mathbf{I}\omega)(v, w) = \omega(\mathbf{I}(v), \mathbf{I}(w)) = \omega(I^2(v), I(w)) = (I(v), w) = \omega(v, w)$$

hence $\mathbf{I}(\omega) = \omega$. This means that $\omega \in \bigwedge^{1,1} V$ because $\mathbf{I}$ acts as i^{p-q} on elements in $\bigwedge^{p,q} V^*$.

Note that two of the three structures $\{\langle , \rangle, I, \omega\}$ determine the other two

Lemma 4.22. *Let $(V, \langle , \rangle)$ be a euclidean vector space with a compatible almost complex structure I. Then $(,) = \langle , \rangle - i\omega$ is a positive definite hermitian form on (V, I).*

Proof. For $v \in V$ we have $(v, v) = \langle v, v \rangle - i\langle I(v), v \rangle$. Because I is compatible $\langle I(v), v \rangle = 0$ which kills the complex part of $(,)$. Therefore $(v, v) = \langle v, v \rangle \geq 0$ because $\langle , \rangle$ is positive definitive. Next, given $v, w \in V$ we have $(v, w) = \langle v, w \rangle - i\langle I(v), w \rangle$.

Can we also extended the form $\langle , \rangle$ to be a positive definite hermitian form on $V_{\mathbb{C}}$ by setting $\langle v \otimes z_v, w \otimes z_w \rangle_{\mathbb{C}} = (z_v \overline{z_w} \langle v, w \rangle$ and extending this bilinearly on V

Lemma 4.23. *Let V be a euclidean vector space with bilinear form $\langle\rangle$. If I is a compatible almost complex structure then $\langle\rangle_{\mathbb{C}}$ is orthogonal with respect to the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$*

Proof. From lemma 4.6 we know that an arbitrary element of $V^{1,0}$ has the form $\frac{1}{2}(v - iI(v))$ for some $v \in V$. Similarly an arbitrary element of $V^{0,1}$ looks like $\frac{1}{2}(w + iI(w))$ for some $w \in V$. Then using the definition of $\langle,\rangle_{\mathbb{C}}$ we compute

$$\langle v - iI(v), w + iI(w)\rangle_{\mathbb{C}} = \langle v, w\rangle_{\mathbb{C}} - i\langle v, I(w)\rangle_{\mathbb{C}} - i\langle I(v), w\rangle_{\mathbb{C}} - \langle I(v), I(w)\rangle_{\mathbb{C}} =$$

$$\langle v, w\rangle_{\mathbb{C}} - i\langle v, I(w)\rangle_{\mathbb{C}}) + i\langle v, I(w)\rangle_{\mathbb{C}} - \langle v, w\rangle_{\mathbb{C}} = 0$$

Note that the second to last equality holds because we have a compatible almost complex structure, so that $\langle I(v), w\rangle_{\mathbb{C}} = \langle I^2(v), I(w)\rangle_{\mathbb{C}} = -\langle v, I(w)\rangle_{\mathbb{C}}$

$\square$

Lemma 4.24. *Let V be a euclidean vector space with bilinear form $\langle,\rangle$ and a compatible almost complex complex structure I. Then under the isomorphism $(V, I) \cong (V^{1,0}, i)$ we have $\frac{1}{2}(,) = \langle,\rangle_{\mathbb{C}|V^{1,0}}$*

Proof. As above we have an isomorphism $(V, I) \cong (V^{1,0}, i)$ given by $v \mapsto \frac{1}{2}(v - iI(v))$. Then for arbitrary $v, w \in V$ we compute

$$\langle\frac{1}{2}(v - iI(v)), \frac{1}{2}(w - iI(w))\rangle_{\mathbb{C}} = \frac{1}{4}\langle v - iI(v), w - iI(w)\rangle_{\mathbb{C}} =$$

$$\frac{1}{4}(\langle v, w\rangle_{\mathbb{C}} + i\langle v, I(w)\rangle_{\mathbb{C}} - i\langle I(v), w\rangle_{\mathbb{C}} + \langle v, w\rangle_{\mathbb{C}}) =$$

$$\frac{1}{4}(2\langle v, w\rangle_{\mathbb{C}} + 2i\langle v, I(w)\rangle_{\mathbb{C}}) = \frac{1}{2}(\langle v, w\rangle_{\mathbb{C}} - i\omega(v, w)) = \frac{1}{2}(v, w)$$

$\square$

Proposition 4.25. *If* $x_1, .., x_k \in V^{1,0}$ *with* $y_i = I(x_i)$ *be such that* $\{x_1, y_1, x_2, y_2, ..., x_k, y_k\}$ *is an orthonormal basis for* V *with respect to* $\langle , \rangle$. *Then we have*

$$\omega = \frac{i}{2}\Sigma_{i=1}^k z^i \wedge \overline{z^i} = \Sigma_{i=1}^k x^i \wedge y^i$$

with $z_i = \frac{1}{2}(x_i - iy_i)$ *and superscripts denoting the dual basis*

Proof. First note that the z_i form a $\mathbb{C}$-basis for $V^{1,0}$. From the lemma we have that the hermitian form $\langle , \rangle_{\mathbb{C}}$ is given by a hermitian matrix $\frac{1}{2}(h_{ij})$ where

$$\langle \Sigma_{i=1}^k a_i z_i, \Sigma_{j=1}^k b_j z_j \rangle = \frac{1}{2}\Sigma_{i,j=1}^n h_{i,j} a_i \overline{b_j}$$

Note that the lemma also gives $(x_i, x_j) = 2\langle x_i, x_j \rangle_{\mathbb{C}} = 2\langle z_i, z_j \rangle_{\mathbb{C}} = h_{i,j}$. Since the form $\langle , \rangle_{\mathbb{C}}$ is hermitian we also have

$$(x_i, y_j) = (x_i, I(x_j)) = (x_i, i \cdot x_i) = -i(x_i, x_j) = -ih_{i,j}$$

and

$$(y_i, y_j) = (I(x_i), I(x_j)) = (i \cdot x_i, i \cdot x_j) = -i^2(x_i, x_j) = h_{i,j}$$

Now using the above and the definition of $(,)$ we see that

$$\omega(x_i, x_j) = \omega(y_i, y_j) = -Im(h_{i,j})$$

$$\omega(x_i, y_j) = Re(h_{i,j})$$

$$\langle x_i, x_j \rangle = \langle y_i, y_j \rangle = Re(h_{i,j})$$

$$\langle x_i, y_j \rangle = Im(h_{i,j})$$

This means that we must have

$$\omega = -\Sigma_{i<j} Im(h_{i,j})(x^i \wedge x^j + y^i \wedge x^j) + \Sigma_{i,j=1}^n Re(h_{i,j})(x^i \wedge y^j)$$

Now suppose the basis $\{x_1, y_1, x_2, y_2, ..., x_k, y_k\}$ is orthonormal with respect to $\langle,\rangle$. Then from the above description of $\langle,\rangle$ on the aforementioned basis we must have $h_{i,j} = 0$ for $i \neq j$ and $h_{i,i} = 1$ for each i with $1 \leq i \leq n$. This shows that

$$\omega = \Sigma_{i=1}^n x^i \wedge y^i$$

Also using

$$z^i \wedge \overline{z}^j = (x^i + iy^i) \wedge (x^j - iy^j) = x^i \wedge x^j - i(x^i \wedge y^j + x^j \wedge y^i) + y^i \wedge y^j$$

gives the equality

$$\omega = \frac{i}{2}\Sigma_{i=1}^k z^i \wedge \overline{z^i}$$

$\square$

The above proposition is useful because there always exists an orthonormal basis of V with respect to $\langle,\rangle$, so long as we have a compatible almost complex structure I. To see this note that we can pick some $x_1 \neq 0$ such that $\langle,\rangle$. Such an x_1 always exists, this is because $(V, \langle,\rangle)$ is euclidean so there is an isomorphism ϕ from V to $\mathbb{R}^n$ that respects $\langle,\rangle$. This means we can set $x_1 = I(e_1)$ to get the required x_1. After x_1 is chosen it is automatically orthogonal to $y_1 = I(x_1)$ since I is a compatible almost complex structure

Definition 4.26. If $(V, \langle,\rangle)$ is a euclidean vector space with an almost compatible structure I then the Lefschetz operator $L : \wedge^* V_{\mathbb{C}}^* \mapsto \wedge^* V_{\mathbb{C}}^*$ is defined by $\alpha \mapsto \omega \wedge \alpha$. Where ω is the fundamental form as defined above

Proposition 4.27. (1) *L is the $\mathbb{C}$-linear extension of the real operator $\wedge^* V^* \mapsto V^*$ given by $\alpha \mapsto \omega \wedge \alpha$*

(2) *The Lefschetz operator has bidegree $(1,1)$, meaning we have*

$$L(\bigwedge^{p,q} V^*) \subset \bigwedge^{p+1,q+1} V^*$$

(3) *For each k the map $L^k : \wedge^k V^* \mapsto \wedge^{2n-k} V^*$ is a bijection*

Proof.

(1) This follows from $(\bigwedge^* V^*)_{\mathbb{C}} = \bigwedge^* V^*_{\mathbb{C}}$

(2) This is a consequence of proposition 4.25 since we can choose an orthonormal basis so that

$$\omega = \Sigma_{i=1}^n x^i \wedge y^i$$

Then if $\alpha \in \bigwedge^{p,q} V^*$ we have $(x^i \wedge y^i) \wedge \alpha \in \bigwedge^{p+1,q+1} V^*$ for each i. This shows that $\omega \wedge \alpha$ is a sum of elements in $\bigwedge^{p+1,q+1} V^*$

□

Definition 4.28. Let $(V, \langle,\rangle)$ be an oriented euclidean vector space of dimension d. Then for any k there is a scalar product on $\bigwedge^k V$ give by

$$\langle v_1 \wedge ... \wedge v_n, w_1 \wedge ... \wedge w_n \rangle = detA$$

where $A_{i,j} = \langle v_i, w_j \rangle$

We can use the same method to define a scalar product on $\bigwedge^k V^*$. We just need scalar product on the dual space V^* and this is given by defining $\langle e^i, e^j \rangle = \langle e_i, e_j \rangle$ where the e^i denote the dual basis

Definition 4.29. Let $(V, \langle , \rangle)$ be a euclidean vector space of dimension d. Also let V have orientation $vol = e_1 \wedge ... \wedge e_d$ for some basis $e_1, ..., e_d$. Then the Hodge star operator $* : \bigwedge^k V \mapsto \bigwedge^{d-k} V$ is defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot vol$$

where the above holds for all $\alpha \in \bigwedge^k V$

In order for the above definition to make sense we need to prove $*\beta$ is well defined given any $\beta \in \bigwedge^k V$. One way to see this is to choose an orthonormal basis $e_1, .., e_d$ of V, so that

$$\{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k} : 1 \leq i_1 < i_2 < ... < i_k \leq d - k\}$$

is an orthonormal $\mathbb{R}$-basis for $\bigwedge^k V$. Hence if $*\beta \in \bigwedge^{d-k} V$ satisfies the above condition we can write $*\beta = \Sigma_{r=1}^{\binom{d}{k}} \beta_r e_{I_r}$ Where we define the e_{I_r} via the identification

$$\{e_{I_j} | 1 \leq j \leq \binom{d}{k}\} = \{e_{i_1} \wedge ... \wedge e_{i_{d-k}} | 1 \leq i_1 < ... < i_{d-k} \leq d\}$$

Then given any $e_{j_1} \wedge ... \wedge e_{j_k} \in \bigwedge^k V$ where $1 \leq j_1 < ... < j_k \leq d$ we have

$$e_{j_1} \wedge ... \wedge e_{j_k} \wedge *\beta = e_{j_1} \wedge ... \wedge e_{j_k} \wedge (\sum_{r=1}^{\binom{d}{k}} \beta_r e_{I_r}) = \sum_{r=1}^{\binom{d}{k}} \beta_r \cdot e_{j_1} \wedge ... \wedge e_{j_k} \wedge e_{I_r}$$

However, note that $e_{j_1} \wedge ... \wedge e_{j_k} \wedge e_{I_r} \neq 0$ only if $e_{I_r} = e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_{d-k}}$ where $\{i_1, i_2, ..., i_{d-k}\} = \{1, 2, ..., d\} - \{j_1, j_2, ..., j_k\}$. Hence from the above

$$e_{j_1} \wedge ... \wedge e_{j_k} \wedge *\beta = e_{j_1} \wedge ... \wedge e_{j_k} \wedge e_{i_1} \wedge ... \wedge e_{i_{d-k}} = \beta_q sgn(j_1, ..., j_k, i_1, ..., i_{d-k}) \cdot vol$$

for some $q \in \{1, ..., \binom{d}{k}\}$ then by definition of the Hodge star operator we have

$$\beta_q sgn(j_1, ..., j_k, i_1, ..., i_{d-k}) \cdot vol = \langle e_{j_1} \wedge ... \wedge e_{j_k}, \beta \rangle \cdot vol$$

which implies that $\beta_q = sgn(j_1, ..., j_k, i_1, ..., i_{d-k}) \langle e_{j_1} \wedge ... \wedge e_{j_k}, \beta \rangle$. This shows that any $*\beta$ satisfying the condition of definition 2.20 has unique coefficients under a given basis. Therefore such a $*\beta$ must be unique. It should also be noted that there is a Hodge star operator on $\bigwedge^k V^*$ defined by using the definition above with the scalar product on V^*

Proposition 4.30. *Let $(V, \langle , \rangle)$ be an euclidean vector space of dimension d with orientation as above. Furthermore suppose that the basis $e_1, ..., e_d$ is orthonormal. Then the associated Hodge start operator satisfies the following:*

(1) *For $\{i_1, ..., i_k, j_1, ..., j_{d-k}\} = \{1, ..., d\}$ we have*

$$*(e_{i_1} \wedge ... \wedge e_{i_k}) = \lambda \cdot e_{j_1} \wedge ... \wedge e_{j_{d-k}}$$

where $\lambda = sgn(i_1, ..., i_k, j_1, ..., j_{d-k})$. Note that $*1 = vol$ where $1 \in \bigwedge^0 V = \mathbb{R}$.

(2) *For $\alpha \in \bigwedge^k V$ and $\beta \in \bigwedge^{d-k} V$ we have*

$$\langle *\alpha, \beta \rangle = (-1)^{k(d-k)} \langle \alpha, *\beta \rangle$$

(3) *The Hodge star operator is an isometry on $(\bigwedge^* V, \langle , \rangle)$*

(4)

$$(*|_{\bigwedge^k V})^2 = (-1)^{k(d-k)}$$

Proof. To prove (1), note that from the definition of the Hodge star operator and the fact that $\{e_{i_1} \wedge ... \wedge e_{i_k} : 1 \leq i_1 < i_2 < ... < i_k \leq d-k\}$ is an orthonormal basis gives

$$e_{i_1} \wedge ... \wedge e_{i_k} \wedge *(e_{i_1} \wedge ... \wedge e_{i_k}) = vol$$

This implies that

$$*(e_{i_1} \wedge ... \wedge e_{i_k}) = \lambda \cdot e_{j_1} \wedge ... \wedge e_{j_{d-k}}$$

where $\{i_1, ..., i_k, j_1, ..., j_{d-k}\} = \{1, ..., d\}$ and $\lambda = sgn(i_1, ..., i_k, j_1, ..., j_{d-k})$. Next, note that in For $\alpha \in \bigwedge^k V$ and $\beta \in \bigwedge^{d-k} V$

$$\langle *\alpha, \beta\rangle \cdot vol = *\alpha \wedge *\beta = \alpha_1 \wedge ... \wedge \alpha_{d-k} \wedge \beta_1 \wedge ... \wedge \beta_k =$$

$$(-1)^{d-k}(\beta_1 \wedge \alpha_1 \wedge ... \wedge \alpha_{d-k} \wedge \beta_2 \wedge ... \wedge \beta_k) =$$

$$(-1)^{k(d-k)}(\beta_1 \wedge ... \wedge \beta_k \wedge \alpha_1 \wedge ... \wedge \alpha_{d-k}) =$$

$$(-1)^{k(d-k)}(*\beta \wedge *\alpha) = (-1)^{k(d-k)}\langle *\beta, \alpha\rangle \cdot vol$$

But this implies that $\langle *\alpha, \beta\rangle = (-1)^{k(d-k)}\langle \alpha, *\beta\rangle$ proving part (2). To prove part (3) we will show that $*$ is an isometry on the basis $\mathbf{B} = \{e_{i_1} \wedge ... \wedge e_{i_k} : 1 \leq i_1 < i_2 < ... < i_k \leq k\}$. Suppose that $e_I, e_J \in \mathbf{B}$ with $I \neq J$, then we must have $\{1, ..., d\} - I \neq \{1, ..., d\} - J$ hence $*e_I \neq *e_j$. But then from a and the fact that $\{e_{i_1} \wedge ... \wedge e_{i_k} : 1 \leq i_1 < i_2 < ... < i_k \leq d-k\}$ is an orthonormal basis of $\bigwedge^* V^{d-k}$ it follows that $\langle *e_I, *e_J\rangle = 0 = \langle e_I, e_J\rangle$. Also from (1) we have $\langle *e_I, *e_I\rangle = \langle \lambda e_I', \lambda e_I'\rangle$ where $\lambda = sgn(i_1, ..., i_k, j_1, ..., j_{d-k})$ and $I' = \{1, ..., d\} - I$. But $\lambda = \pm 1$ so that $\lambda^2 = 1$ and we have

$$\langle \lambda e_I', \lambda e_I'\rangle = \lambda^2 \langle e_I', e_I'\rangle = \langle e_I', e_I'\rangle = 1$$

. This gives $\langle *e_I, *e_I\rangle = \langle e_I, e_I\rangle = 1$ finishing the proof of c. Next let $\beta \in \bigwedge^k V$ by arbitrary. Then for all $\alpha \in$ using (2) and (3) we have

$$\langle \alpha, \beta\rangle = \langle *\alpha, *\beta\rangle = (-1)^{k(d-k)}\langle \alpha, **\beta\rangle = \langle \alpha, (-1)^{k(d-k)} **\beta\rangle$$

but since the form $\langle , \rangle$ is non degenerate this gives $**\beta = (-1)^{k(d-k)}\beta$ proving (4) $\square$

Definition 4.31. The dual Lefschetz operator $\Lambda : \bigwedge^* V^* \mapsto \bigwedge^* V^*$ is uniquely determined by

$$\langle \Lambda\alpha, \beta\rangle = \langle \alpha, L\beta\rangle$$

for all $\beta \in \bigwedge^* V^*$. The $\mathbb{C}$-linear extension on $\bigwedge^* V^*_{\mathbb{C}}$ of the dual operator is also denoted Λ.

Proposition 4.32. *If $\{x_1, y_1, ..., x_n, y_n\}$ is an orthonormal basis for V as in proposition 4.25 then*

$$\omega^n = n! \cdot vol$$

where $vol = x^1 \wedge y^1 \wedge ... \wedge x^n \wedge y^n \in \bigwedge^n V^*$

Proof. From proposition 4.25 we have

$$\omega = \sum_{i=1}^{k} x^i \wedge y^i$$

so that

$$\omega^n = (\sum_{i=1}^{k} x^i \wedge y^i)^n = \sum_{\sigma \in S_n} (\sum_{i=1}^{n} x^{\sigma(i)} \wedge y^{\sigma(i)})$$

However, note that for all $\sigma \in S_k$ we have

$$\sum_{i=1}^{k} x^{\sigma(i)} \wedge y^{\sigma(i)} = sgn(\sigma)^2 \sum_{i=1}^{n} x^i \wedge y^i = \sum_{i=1}^{k} x^i \wedge y^i = vol$$

hence the above gives

$$\omega^n = |S_n| \cdot vol = n! \cdot vol$$

□

Proposition 4.33. *The operator* Λ *is of degree* -2*, meaning* $\Lambda(\bigwedge^k V^*) \subset \bigwedge^{k-2} V^*$*. Furthermore we have*

$$\Lambda = *^{-1} \circ L \circ *$$

Proof. Given $\alpha \in \bigwedge^* V^k$ we consider an arbitrary $\beta \in \bigwedge^* V^*$. Then we have

$$\langle \alpha, L\beta \rangle \cdot vol = \langle L\beta, \alpha \rangle \cdot vol = L\beta \wedge *\alpha =$$

$$\omega \wedge \beta \wedge *\alpha = \beta \wedge \omega \wedge *\alpha$$

where the last equality follows from proposition 4.25 and the fact that if $v \in \bigwedge^k V$ then

$$x_i \wedge y_i \wedge v = (-1)^k x_i \wedge v \wedge y_i = (-1)^2 kv \wedge x_i \wedge y_i = v \wedge x_i \wedge y_i$$

However, this implies that

$$\langle \alpha, L\beta \rangle \cdot vol = \beta \wedge (L(*\alpha) =$$

$$\langle \beta, *^{-1} \circ L \circ *(\alpha)) \cdot vol = \langle *^{-1} \circ L \circ *(\alpha)), \beta \rangle \cdot vol$$

showing that $\langle \alpha, L\beta \rangle = \langle *^{-1} \circ L \circ *(\alpha)), \beta \rangle$ for all $\beta \in \bigwedge^* V^*$ completing the proof.

□

Proposition 4.34. *With* $\langle , \rangle_{\mathbb{C}}$, Λ, *and* $*$ *as defined previously we have*

(1) *The decomposition* $\bigwedge^k V^*_{\mathbb{C}} = \bigoplus \bigwedge^{p,q} V^*$ *is orthogonal with respect to* $\langle , \rangle_{\mathbb{C}}$.

(2) *If* $n = dim_{\mathbb{C}}(V, I)$ *then*

$$*(\bigwedge^{p,q} V^*) \subset \bigwedge^{n-q,n-q} V^*$$

(3)

$$\Lambda(\bigwedge^{p,q} V^*) \subset \bigwedge^{p-1,q-1} V^*$$

Definition 4.35. The map $H : \bigwedge^* V \mapsto \bigwedge^* V$ defined by

$$H = \Sigma_{k=0}^{2n}(k-n) \cdot \Pi^k$$

in other words H is the map that is multiplication by $k - n$ when restricted to $\bigwedge^k V$. An analogous operator is defined on $\bigwedge^* V^*$ and will also be called H

Proposition 4.36. *For the operators* L, Λ, H *defined above me have*

(1)

$$[H, L] = 2L$$

(2)

$$[H, \Lambda] = -2\Lambda$$

(3)

$$[L, \Lambda] = H$$

where $[A, B] = AB - BA$

Proof. (1) The vector space V has an almost complex structure so by proposition dim it must have dimension $d = 2n$ for some integer n. Given $\alpha \in \bigwedge^* V*$ from proposition J we must have $\alpha = \sum_{i=0}^{2n} \alpha_i v_i$

where $v_i \in \bigwedge^i V^*$ for each i. Then we have

$$HL(\alpha) = HL(\sum_{i=0}^{2n} \alpha_i v_i) = H(\sum_{i=0}^{2n} L(\alpha_i v_i)) =$$

$$\sum_{i=0}^{2n}(i-n+2)L(\alpha_i v_i) = (i-n+2)L(\alpha)$$

where the second equality follows because $L(\bigwedge^k V^*) \subset \bigwedge^{k+2} V^*$. On the other hand

$$LH(\alpha) = LH(\sum_{i=0}^{2n} \alpha_i v_i) = L(\sum_{i=0}^{2n}(i-n)\alpha_i v_i = (i-n)L(\alpha)$$

but then it follows that

$$HL - LH(\alpha) = (i-n+2)L(\alpha) - (i-n)L(\alpha) = 2L(\alpha)$$

completing the proof on a. The proof on b is the same except with L replaced by Λ. The only other difference is that we use proposition R to conclude that $H\Lambda(\alpha) = (i-n-2)\Lambda(\alpha)$. The proof of c relies on a fairly lengthy induction argument and is therefore omitted see [Huy05, p. 34]

$\square$

Proposition 4.37. *Let $(V, \langle\rangle, I)$ be a euclidean vector space of dimension $2n$ with a compatible almost complex structure. Then there is an $sl(2)$-representation on $\bigwedge^* V^*$ where $sl(2)$ is the $\mathbb{R}$-vector space of $2x2$ matrices with trace* 0

Proof. The vector space $sl(2)$ is generated by the elements

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It can also be seen that

$$[B, X] = 2X, [B, Y] = -2Y, [X, Y] = B$$

which shows that [,] defines a function $sl(2) \times sl(2) \mapsto sl(2)$ so that $sl(2)$ is a Lie algebra. Furthermore, we can obtain a linear map from $sl(2)$ into $End(\bigwedge^* V^*)$ by sending $X \mapsto L$, $Y \mapsto \Lambda$, $B \mapsto H$. This map is a lie algebra homomorphism from proposition 4.36. $\square$

Lemma 4.38. *Let $\phi : sl(2) \to GL(V)$ be a representation on a finite dimensional nontrivial complex vector space. Furthermore let $e = \phi(X)$, $f = \phi(Y)$, $g = \phi(B)$, then the following hold*

(1) *V contains a nonzero eigenvector for g.*
(2) *If v is an eigenvector for g with eigenvalue λ then*

$$ge^i(v) = (\lambda + 2i)e^i(v)$$

(3) *If $v \in V$ is a nonzero eigenvector for g with eigenvalue λ then*

$$gf^i(v) = (\lambda - 2i) \cdot f^i(v)$$

(4) *There is a nonzero eigenvector of g in the kernel of f*
(5) *If $i > 0$ and v is an eigenvector of g with eigenvalue λ*

$$[e^i, f](v) = i(\lambda + i - 1)e^{i-1}(v)$$

(6) *If v is a nonzero eigenvector for g in the kernel of f, then*

$$R_v = Span\{e^i(v)\}_{i=0}^{\infty}$$

defines a sub-representation. Furthermore, the irreducible representations are all of this form.

Proof. (1) The characteristic polynomial for g has a root in $\mathbb{C}$ by the fundemental theorem of algebra. Since V is a complex vector space this means there exists an eigenvector $v \in V$ for g with eigenvalue λ. Furthermore, since $dimV \neq 0$ and the eigenspaces of g decompose V we may assume there is a nonzero eigenvector.

(2) We will show inductively that $gf^i(v) = (\lambda + 2i) \cdot f^i(v)$. For $i = 0$ the statement holds from the assumption that v is an eigenvector. Then using induction we have

$$ge^i(v) = ge(e^{i-1}(v) = eg(e^{i-1}(v))+2e(e^{i-1}(v)) = (\lambda+2(i-1))e^i(v)+2e^i(v) = (\lambda+2i)e^i(v)$$

(3) This is true for $i = 1$, since the fact that we have an $sl(2)$ implies

$$[g, f](v) = -2f(v)$$

and implies that

$$gf(v) = fg(v) - 2f(v) = f(\lambda v) - 2f(v) = (\lambda - 2) \cdot f(v)$$

Now we use the lie bracket relations again to show that

$$[g, f](f^{i-1}(v)) = -2f^i(v)$$

for $i > 1$ hence

$$gf^i(v) = fg(f^{i-1}(v)) - 2f^i(v) = f((\lambda - 2(i-1)) \cdot f^{i-1}(v)) - 2f^i(v)$$

$$= (\lambda - 2i) \cdot f^i(v)$$

(4) From (3) each vector $f^i(v)$ is an eigenvector of g with eigenvalue $\lambda - 2i$. Hence if there was no j such that $f^j(v) = 0$ we would

obtain an infinite sequence of nonzero eigenvectors for f with distinct eigenvalues. This is a contradiction to the assumption that V is finite dimensional, since eigenvectors with distinct eigenvalues are linearly independent. Hence we may take $j \in \mathbb{N}$ with j minimal such that $f^j(v) = 0$. Then we have $f^{j-1}(v) \neq 0$ which verifies the existence of a nonzero eigenvector for g in the kernel of f.

(5) we use induction on i. The statement is true for $i = 1$ since using the lie bracket relations we have

$$[e, f](v) = g(v) = \lambda \cdot v$$

then by induction

$$[e^i, f](v) = e^i f(v) - f e^i(v) = e(e^{i-1} f(v) - f e^{i-1}(v)) + (e f e^{i-1}(v) - f e^i(v)) =$$

$$e([e^{i-1}, f](v)) + [e, f](e^{i-1}(v)) = e((i-1)(\lambda + (i-1) - 1)e^{i-2}(v) + g e^{i-1}(v) =$$

$$(i-1)(\lambda + (i-1) - 1)e^{i-1}(v) + (2(i-1) + k - n)e^{i-1}(v) =$$

$$((i-1)\lambda + \lambda + (i-1)((i-1) - 1 + 2))e^{i-1}(v) = i(\lambda + i - 1)e^{i-1}(v)$$

(6) The fact that R_v is a sub-representation is a consequence of (2) and (5). If W in any irreducible representation then it is an $sl(2)$-representation so there is a nonzero eigenvector $v \in W$ in the kernel of g. But then R_v is a sub-representation of W which means that $R_v = W$ since W is irreducible.

□

Definition 4.39. A primitive element of $\bigwedge^* V^*$ is an $\alpha \in \bigwedge^k V^*$ such that $\Lambda\alpha = 0$. The subspace of all primitive elements in $\bigwedge^k V$ is denoted P^k. The subspace of primitive elements in $\bigwedge^* V^*_{\mathbb{C}}$ is the complexication of $P^k_{\mathbb{C}}$

Lemma 4.40. *If $i \neq j$ then the subspaces $L^i P^{k-2i}_{\mathbb{C}}$ and $L^j P^{k-2j}_{\mathbb{C}}$ of $\bigwedge^* V^*$ are orthogonal*

Proof. Suppose $\alpha \in P^{k-2i}$ and $\beta \in P^{k-2j}$ and assume $WLOG$ that $i < j$. Using proposition 4.37 we have an $sl(2)$ representation ϕ on $\bigwedge^* V^*_{\mathbb{C}}$ with $\phi(X) = L$, $\phi(Y) = \Lambda$, and $\phi(B) = H$. Then applying lemma 4.38 part we see that if $v \in \bigwedge^* V^* \subset \bigwedge^* V^*_{\mathbb{C}}$ is an eigenvector for H with eigenvalue λ then for $m > 0$

$$(*) \qquad [L^m, \Lambda](v) = m(\lambda + m - 1)L^{m-1}(v)$$

But in view of definition 4.35 one can see that v in an eigenvalue for H iff $v \in \bigwedge^l V^*$ for some positive integer l. In particular $(*)$ holds when v is a primitive vector and therefore using $\Lambda v = 0$ gives

$$\Lambda L^m(v) = -[L^m, \Lambda](v) = m(1 - m - \lambda)L^{m-1}(v)$$

and applying this inductively and using the linearity of Λ we see

$$(**) \qquad \Lambda^i L^i(\alpha) = c\dot{\alpha}$$

for some constant c. But then since $i < j$ it makes sense to use definition 4.31 $i + 1$ times to see that

$$\langle L^i(\alpha), L^j(\beta)\rangle = \langle \Lambda(\Lambda^i L^i(\alpha)), L^{j-i-1}(\beta)\rangle = \langle \Lambda(c\cdot\alpha), L^{j-i-1}(\beta)\rangle = \langle 0, L^{j-i-1}(\beta)\rangle = 0$$

$\square$

Proposition 4.41. *If $(V, \langle , \rangle)$ is a euclidean vector space with a compatible almost complex structure then*

$$\bigwedge^k V^k = \bigoplus_{i \geq 0} L^i(P^{k-2i})$$

and this decomposition is orthogonal with respect to $\langle , \rangle$ on $\bigwedge^k V^k$.

Proof. The subspaces $L^i P^{k-2i} \subset L^i P^{k-2i}_{\mathbb{C}}$ and $L^j P^{k-2j} \subset L^j P^{k-2j}_{\mathbb{C}}$ are orthogonal subspaces of $\bigwedge^k V^*$ from 4.40 which gives

$$\bigoplus_{i \geq 0} L^i(P^{k-2i}) \subseteq \bigwedge^k V^*$$

so the only part of the proposition that needs proof is that these subspaces span $\bigwedge^k V^*$. Applying lemma 4.38 part (iii) to the $sl(2)$-representation on $\bigwedge^* V^*_{\mathbb{C}}$ we see that there exists a nonzero eigenvector for H in the kernel of Λ. But we have v is an eigenvector for H iff $v \in \bigwedge^l V^*$ for some positive integer l. Then in view of definition 4.39 the above shows that $\bigwedge^* V^*$ admits a primitive vector. Furthermore using part (vi) of the lemma we see that the irreducible subrepresentations of $\bigwedge^* V^*_{\mathbb{C}}$ all have the form $R_v = span\{v, Lv, L^2 v, ...\}$. Since any finite $sl(2)$ representation is a direct sum of irreducible representations we have

$$\bigwedge^* V^*_{\mathbb{C}} = \bigoplus_{v \in \mathcal{A}} R_v$$

where $\mathcal{A}$ is some appropriate indexing set. Then given arbitrary $w \in \bigwedge^k V^* \subset \bigwedge^* V^*$ for some indexing set $\mathcal{B}$ we have

$$w = \sum_{r \in \mathcal{B}} a_r L^{i_r} v_r$$

where v_r is primitive for each r. For each r let $d_r = deg(v_r) + 2i_r$ where the degree of a primitive vector v is equal to l if $v \in \bigwedge^l V^*$. Then we can rewrite the above sum as

$$w = \sum_{j=0}^{2n} \Big(\sum_{r:d_r=j} a_r L^{i_r} v_r \Big)$$

But since w is in wedge k and we have a decomposition

$$\bigwedge^* V^* = \bigoplus_{j=0}^{2n} \bigwedge^j V_{\mathbb{C}}^*$$

we must have $a_r \neq 0$ iff $d_r = k$. In other words we have

$$w = \sum_{r:d_r=k} a_r L^{i_r} v_r$$

which shows that the subspaces $L^i P_{\mathbb{C}}^{k-2i}$ span $\bigwedge^k V_{\mathbb{C}}^*$ which gives the decomposition

$$\bigwedge^k V_{\mathbb{C}}^k = L^i (P^{k-2i})_{\mathbb{C}}$$

the statement of the proposition follows by looking at the real part. $\square$

Lemma 4.42. *For all $\alpha \in P^k$ we have*

$$*L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} \cdot L^{n-k-j} \mathbf{I}(\alpha)$$

Proof. For the proof see [Huy05, p. 37] $\square$

Definition 4.43. Let $(V, \langle,\rangle, I)$ be as above with ω the fundamental form. Then there is a bilinear form Q on $\bigwedge^k V^*$ defined by

$$(\alpha, \beta) \mapsto (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k}$$

Proposition 4.44.

(1)

$$Q(\bigwedge^{p,q} V^*, \bigwedge^{p',q'} V^*) = 0$$

for $(p,q) \neq (q',p')$

(2)

$$i^{p-q} Q(\alpha, \overline{\alpha}) = (n - (p+q))! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}} > 0$$

with $0 \neq \alpha \in P^{p,q}$ *and* $p + q \leq n$

Proof. For (1) if $\alpha \in \bigwedge^{p,q} V^*$, and $\beta \in \bigwedge^{p',q'} V^*$ with $(p,q) \neq (q',p')$ then we may assume WLOG that $p > q'$ so that $p' + q' = k$ implies $p + p' > k$. Note that $\alpha \wedge \beta \wedge \omega^{n-k}$ has type $(p + p' + n - k, q + q' + n - k)$ with $p + p' + n - k > k + n - k = n$ but $\bigwedge^l (V^*)^{1,0}$ is trivial for $l > n$ because $(V^*)^{1,0}$ has dimension n. This implies that $\alpha \wedge \beta \wedge \omega^{n-k}$ is zero.

For (2) we have

$$Q(\alpha, \overline{\alpha}) \cdot vol = (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \overline{\alpha} \wedge \omega^{n-k} =$$

$$(-1)^{\frac{k(k-1)}{2}} \alpha \wedge L^{n-k}(\overline{\alpha}) = (-1)^{\frac{k(k-1)}{2}} \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot vol$$

For some $\beta \in \bigwedge^k V^*$ with $*\overline{\beta} = L^{n-k}\overline{\alpha}$. From 4.30 part (d) we have

$$(*|_{\bigwedge^k V})^2 = (-1)^{k(2n-k)}$$

If k is odd (respectively even) then $k(2n-k)$ is odd (respectively even) so that the above becomes

$$(*|_{\bigwedge^k V})^2 = (-1)^k$$

and we have $*^2\overline{\beta} = (-1)^k \overline{\beta}$. But using lemma 4.42 gives

$$\overline{\beta} = (-1)^k *^2 \overline{\beta} = (-1)^k * L^{n-k}(\overline{\alpha}) = (-1)^{\frac{k(k+1)}{2} + k} (n-k)! i^{q-p} \overline{\alpha}$$

then taking complex conjugates of both sides we see that

$$\beta = (-1)^{\frac{k(k+1)}{2}+k}(n-k)!i^{p-q}\alpha$$

and plugging this into the above gives

$$Q(\alpha,\overline{\alpha}) = (-1)^{\frac{k(k-1)}{2}}\langle \alpha, (-1)^{\frac{k(k+1)}{2}+k}(n-k)!i^{p-q}\cdot\alpha\rangle_{\mathbb{C}}$$

$$= (-1)^{\frac{k(k-1)}{2}+\frac{k(k+1)}{2}+k}(n-k)!i^{q-p}\langle \alpha,\alpha\rangle_{\mathbb{C}} = (n-k)!i^{q-p}\langle \alpha,\alpha\rangle_{\mathbb{C}}$$

Note that

$$\frac{k(k-1)}{2} + \frac{k(k+1)}{2} + k = k(k+1)$$

and $k(k+1)$ is always even since either k or $k+1$ must be even. But then if $0 \neq \alpha \in P^{p,q}$ using $k = p+q$ in the above formula gives

$$i^{p-q}Q(\alpha,\overline{\alpha}) = (n-(p+q))!\cdot\langle \alpha,\alpha\rangle_{\mathbb{C}} > 0$$

$\square$

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