Exercise 6.3

Introduction to Discrete Mathematics
MATH 2001

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ABSTRACT. This is Exercise 6.3 from Hammack [Ham13, Ch. 6]:

Exercise 6.3. Use the method of proof by contradiction to prove the following statement: *The real number $\sqrt[3]{2}$ is irrational.*

Solution. Assume for the sake of contradiction that $\sqrt[3]{2}$ *were* rational. Then, by definition, there would exist integers a and b with $b \neq 0$ such that $\sqrt[3]{2} = \frac{a}{b}$. Cubing both sides of this equation, we would arrive at the equation $2 = \frac{a^3}{b^3}$, and then multiplying both sides of the equation by b^3, we would have

$$2b^3 = a^3.$$

Considering unique factorizations of the integers a and b, we see that the number of powers of 2 in the prime factorization of the right hand side of the equation, i.e., a^3, is divisible by 3, while the number of powers of 2 in the prime factorization of the left hand side of the equation, i.e., $2b^3$, is not divisible by 3, giving a contradiction.

Therefore, the assumption that $\sqrt[3]{2}$ was rational was false, and we have proven that $\sqrt[3]{2}$ is not rational. $\square$

Remark 0.1. In an earlier homework problem we worked with unique factorizations of integers into primes. For convenience, we recall in more detail how this is used in the solution above. We have that a has a unique factorization of the form $a = (-1)^{r_{-1}}2^{r_2}3^{r_3}5^{r_5}7^{r_7}\cdots$, where $r_{-1} \in \{0,1\}$ and $r_2, r_3, r_5, r_7, \ldots$ are non-negative integers, with all but finitely many equal to zero. Similarly, we have that $b = (-1)^{s_{-1}}2^{s_2}3^{s_3}5^{s_5}7^{s_7}\cdots$, where $s_{-1} \in \{0,1\}$ and $s_2, s_3, s_5, s_7, \ldots$ are non-negative integers, with all but finitely many equal to zero. Then we have $a^3 = (-1)^{r_{-1}}2^{3r_2}3^{3r_3}5^{3r_5}7^{3r_7}\cdots$, and $2b^3 = (-1)^{s_{-1}}2^{3s_2+1}3^{3s_3}5^{3s_5}7^{3s_7}\cdots$. Since $a^3 = 2b_3$, then, by unique factorization, we must have that $3r_2 = 3s_2 + 1$; but this is impossible, since $3r_2$ is divisible by 3, whereas $3s_2 + 1$ is not.

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References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

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