Exercise 12.2.10

Introduction to Discrete Mathematics
MATH 2001

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ABSTRACT. This is Exercise 12.2.10 from Hammack [Ham13, §12.2]:

Exercise 12.2.10. Prove the map ("function") $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \left(\frac{x+1}{x-1}\right)^3$ is bijective.

Solution. As we have seen, showing that f is bijective is equivalent to showing that f admits an inverse map; i.e., there exists a map $f^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ such that $f^{-1} \circ f = \mathrm{Id}_{\mathbb{R}-\{1\}}$ and $f \circ f^{-1} = \mathrm{Id}_{\mathbb{R}-\{1\}}$.

To make the algebra a little easier, let us observe that we can view f as the composition of the functions

$$\mathbb{R} - \{1\} \xrightarrow[a(x)=\frac{x+1}{x-1}]{a} \mathbb{R} - \{1\} \xrightarrow[b(x)=x^3]{b} \mathbb{R} - \{1\}, \qquad f = b \circ a$$

The first claim is that b is bijective, with inverse $b^{-1}(x) = x^{1/3}$. Indeed, for all $x \in \mathbb{R} - \{1\}$, we have $b^{-1}(b(x)) = x$ and $b(b^{-1}(x)) = x$. Our next claim is that a is bijective, with inverse $a^{-1}(x) = \frac{1+x}{1-x}$. Indeed, if we set $y = \frac{x+1}{x-1}$, then a little algebra gives us that this is equivalent to $x = \frac{1+y}{1-y}$, so that for all $x \in \mathbb{R} - \{1\}$, we have $a^{-1}(a(x)) = x$ and $a(a^{-1}(x)) = x$.

Finally, the claim is that we may define $f^{-1} = a^{-1} \circ b^{-1}$. Indeed, for all $x \in \mathbb{R} - \{1\}$, we have $f^{-1}(f(x)) = a^{-1}(b^{-1}(b(a(x)))) = a^{-1}(a(x)) = x$, and $f(f^{-1}(x)) = b(a(a^{-1}(b^{-1}(x)))) = b(b^{-1}(x)) = x$. □

Remark 0.1. In the solution above, I wanted to demonstrate how using composition of functions can sometimes make it easier to investigate a difficult function, by breaking it into more manageable pieces. On the other hand, one could have perhaps more briefly noted that if we set $y = \left(\frac{x+1}{x-1}\right)^3$, then a little algebra give us that this is equivalent to $x = \frac{1+y^{1/3}}{1-y^{1/3}}$, so that $f^{-1}(x) = \frac{1+x^{1/3}}{1-x^{1/3}}$.

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Remark 0.2. One could also show directly that the map is both injective and surjective. Indeed, given $y \in \mathbb{R} - \{1\}$, if we set $y = \left(\frac{x+1}{x-1}\right)^3$, then a little algebra give us that this is equivalent to $x = \frac{1+y^{1/3}}{1-y^{1/3}}$, so that $f(\frac{1+y^{1/3}}{1-y^{1/3}}) = y$, showing that f is surjective. Similarly, if $x, x' \in \mathbb{R} - \{1\}$, and $y = f(x) = f(x')$, then a little algebra shows us that $x = \frac{1+y^{1/3}}{1-y^{1/3}}$ and $x' = \frac{1+y^{1/3}}{1-y^{1/3}}$, so that $x = x'$, showing that f is injective.

References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

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