**Exercise 11.4.4**

**Introduction to Discrete Mathematics**
**MATH 2001**

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 11.4.4 from Hammack [Ham13, §11.4]:

**Exercise 11.4.4.** Suppose $P$ is a partition of a set $A$. Define a relation $\sim$ on $A$ by declaring $x \sim y$ if and only if $x, y \in X$ for some $X \in P$.

(1) Prove $\sim$ is an equivalence relation on $A$.
(2) Prove that $P$ is the set of equivalence classes of $\sim$.

*Solution to (1).* Suppose $P$ is a partition of a set $A$, and consider the relation $\sim$ on $A$ given by $x \sim y$ if and only if $x, y \in X$ for some $X \in P$. Our goal is to prove that $\sim$ is an equivalence relation on $A$, i.e., that $\sim$ is reflexive, symmetric, and transitive.

First let us show that $\sim$ is reflexive; i.e., that for all $x \in A$, we have $x \sim x$. For this observe that from the definition of a partition, we have that $A = \bigsqcup_{X \in P} X$. In particular, $A = \bigcup_{X \in P} X$. Therefore, given $x \in A$, there exists $X \in P$ such that $x \in X$, and therefore, we have $x \sim x$.

Next let us show that $\sim$ is symmetric; i.e., that for all $x, y \in A$, if $x \sim y$, then $y \sim x$. So let $x, y \in A$, and assume that $x \sim y$. Then from the definition of $\sim$ there exists $X \in P$ such that $x, y \in X$. Therefore, we also have $y \sim x$ (since $y, x \in X$).

Finally, let us show that $\sim$ is transitive; i.e., that for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$. So let $x, y, z \in A$, and assume that $x \sim y$ and $y \sim z$. Then from the definition of $\sim$, there exists $X \in P$ such that $x, y \in X$, and $Y \in P$ such that $y, z \in Y$. From the definition of a partition, if $X, Y \in P$ and $X \cap Y \neq \emptyset$, then $X = Y$ (i.e., two sets in the partition either have empty intersection, or are equal). Therefore, since $y \in X \cap Y$, we have that $X = Y$. This means that $x, z \in X$, so that $x \sim z$. □

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*Solution to (2).* We want to show that $P$ is equal to the set $A/\sim$ of equivalence classes of $\sim$. Before we start, let us first prove the following statement:

$$\text{(0.1)} \qquad \textit{If } X \in P \textit{ and } x \in X, \textit{ then } [x] = X.$$

The proof is as follows: Assume $X \in P$ and $x \in X$; then we have

$$\begin{aligned}
[x] &:= \{z \in A : z \sim x\} && \text{(Definition of } [x]\text{)} \\
&= \{z \in A : \exists Y \in P,\ z, x \in Y\} && \text{(Definition of } \sim\text{)} \\
&= \{z \in A : z, x \in X\} && (x \in Y \implies Y = X) \\
&= \{z \in A : z \in X\} \\
&= X
\end{aligned}$$

For the third equality, recall that from the definition of a partition, if $X, Y \in P$ and $X \cap Y \neq \emptyset$, then $X = Y$; therefore since $x \in X$, if we also have $x \in Y$, then $Y = X$.

Now that we have proven the claim (0.1) above, let us use it to show that $P \subseteq (A/\sim)$. That is to say, given $X \in P$, we want to show that $X$ is an equivalence class $[x]$ of some element $x \in A$. In fact, from the definition of a partition, we have that $X \neq \emptyset$, so that there exists $x \in X$. It follows from the claim (0.1) that $X = [x]$.

Conversely, let us now show that $(A/\sim) \subseteq P$. That is to say, given an equivalence class $[x]$ for some element $x \in A$, we want to show that $[x] \in P$. Another way to say this is that given $x \in A$, we want to show that there is some $X \in P$ such that $[x] = X$. To get started, recall from the definition of a partition that $A = \bigsqcup_{X \in P} X$, so that in particular, $A = \bigcup_{X \in P} X$. Therefore, given $x \in A$, there exists $X \in P$ such that $x \in X$. From the claim (0.1), we have that $[x] = X$. $\square$

## References

[Ham13] Richard Hammack, *Book of proof*, Creative Commons, 2013.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

*Email address*: `casa@math.colorado.edu`