Final Exam

Abstract Algebra 1

MATH 3140

Fall 2022

Sunday December 11, 2022

UPLOAD THIS COVER SHEET!

NAME: ______________________________

PRACTICE EXAM

Question:	**1**	**2**	**3**	**4**	**5**	**6**	Total
Points:	20	20	20	20	20	20	120
Score:							

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 70 minutes to complete the exam.

1. (20 points) • *Show that for a prime p, the polynomial $x^p + a \in \mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$.*

2. • This problem concerns finite groups of units in commutative rings with $1 \neq 0$.

(a) (10 points) *Show that any finite group of units in an integral domain is cyclic.*

[*Hint*: Use what you know about finite groups of units in a field.]

(b) (10 points) *What if R is any commutative ring with $1 \neq 0$? Is it still true that any finite group of units in R is cyclic?*

[*Hint*: Consider the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$.]

3. • Let R and S be commutative rings with $1 \neq 0$. In this problem we will show that for any ideal $I \subseteq R \times S$, there are ideals $I_R \subseteq R$ and $I_S \subseteq S$ such that $I = I_R \times I_S$, and moreover, we will show that $(R \times S)/I \cong (R/I_R) \times (S/I_S)$.

(a) (2 points) *If $\phi : R \to S$ is a homomorphism and $I_R \subseteq R$ is an ideal, show by example that $\phi(I_R)$ need not be an ideal of S.*

(b) (3 points) *If $\phi : R \to S$ is a* surjective *homomorphism and $I_R \subseteq R$ is an ideal, show that $\phi(I_R)$ is an ideal of S.*

(c) (3 points) The first projection map $\pi_1 : R \times S \to R$, $\pi_1(r, s) = r$, is a homomorphism of rings. *If $I \subseteq R \times S$ is an ideal, show that $I_R := \pi_1(I)$ is an ideal of R.* Similarly, the second projection map $\pi_2 : R \times S \to S$, $\pi_1(r, s) = s$, is a homomorphism of rings. *If $I \subseteq R \times S$ is an ideal, show that $I_S := \pi_2(I)$ is an ideal of S.*

(d) (3 points) *If $I_R \subseteq R$ and $I_S \subseteq S$ are ideals, show that $I_R \times I_S$ is an ideal in $R \times S$.*

(e) (3 points) *If I is an ideal in $R \times S$ and we set $I_R := \pi_1(I)$ and $I_S := \pi_2(I)$, show that $I \subseteq I_R \times I_S$.*

(f) (3 points) *If I is an ideal in $R \times S$ and we set $I_R := \pi_1(I)$ and $I_S := \pi_2(I)$, show that $I = I_R \times I_S$.*

[*Hint:* use that R and S have $1 \neq 0$, and consider $(1,0)I$ and $(0,1)I$ to show that $I \supseteq I_R \times I_S$.]

(g) (3 points) In the notation of the previous problem, *show there is an isomorphism*

$$(R \times S)/I \cong (R/I_R) \times (S/I_S).$$

[*Hint:* Define a homomorphism $\phi : R \times S \to (R/I_R) \times (S/I_S)$.]

4. (20 points) • *Find the degree and a basis for the field extension* $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ *over* $\mathbb{Q}$.

[*Hint:* Find a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$, and then find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{2})$.]

5. (20 points) • *Show that if F, E, and K are fields with $F \leq E \leq K$, then K is algebraic over F if and only if K is algebraic over E, and E is algebraic over F.* (You must *not* assume the extensions are finite.)

6. • **TRUE** or **FALSE**. For this problem, and this problem only, **you do not need to justify your answer**.

(a) (4 points) **TRUE** or **FALSE** (circle one). *There exists a commutative ring with unity that has nonzero zero divisors, and has a quotient ring ("factor ring") that is an integral domain.*

(b) (4 points) **TRUE** or **FALSE** (circle one). *If* F *is a field and* $\phi : F \to F$ *is a ring isomorphism, then* ϕ *is equal to the identity.*

(c) (4 points) **TRUE** or **FALSE** (circle one). *An integral domain of characteristic* 0 *is infinite.*

(d) (4 points) **TRUE** or **FALSE** (circle one). *The remainder of* 7^{122} *when divided by* 11 *is* 5*.*

(e) (4 points) **TRUE** or **FALSE** (circle one). *If* R *is a commutative ring with* $1 \neq 0$*, and* $f(x), g(x) \in R[x]$ *are polynomials of degree two and three respectively, then the degree of* $f(x)g(x)$ *is five.*