CHAPTER 6

A brief introduction to linear algebra

1. Motivation

Recall that a vector in the real plane $\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ can be thought of as simply a point in the plane $v = (x_1, x_2) \in \mathbb{R}^2$. In other words, we think of the vector as being determined by its endpoint, since if we know the endpoint, then we know the vector. We add and scale vectors as follows: if $v = (x_1, x_2)$, $w = (y_1, y_2)$, and $\lambda \in \mathbb{R}$, then

$$v + w = (x_1 + y_1, x_2 + y_2), \quad \lambda v = (\lambda x_1, \lambda x_2).$$

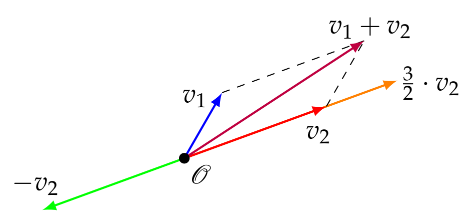

FIGURE 1. Adding and scaling vectors in the plane

Similarly, a vector in real 3-space $\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ can be thought of as simply a point in 3-space $v = (x_1, x_2, x_3) \in \mathbb{R}^3$. We add and scale these vectors in the same way: if $v = (x_1, x_2, x_3)$, $w = (y_1, y_2, y_3)$, and $\lambda \in \mathbb{R}$, then

$$v + w = (x_1 + y_1, x_2 + y_2, x_3 + y_3), \quad \lambda v = (\lambda x_1, \lambda x_2, \lambda x_3).$$

We will use these examples to motivate the definition of an abstract vector space, which informally, is just a collection of things we call vectors, which we can add and scale, and behave, formally, in much the same was as vectors in the plane, or vectors in 3-space.

2. Vector spaces and linear maps

In what follows, fix $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. More generally, K can be any field.

2.1. Vector spaces. Motivated by our intuition of adding and scaling vectors in the plane (see Figure 1), we make the following definition:

Definition 6.2.1 (K-vector space)**.** *A K-**vector space** consists of a triple $(V, +, \cdot)$, where V is a set, and $+ : V \times V \to V$ and $\cdot : K \times V \to V$ are maps, satisfying the following properties:*

(1) *(Group laws)*
   (a) *(Additive identity) There exists an element $\mathscr{O} \in V$ such that for all $v \in V$, $v + \mathscr{O} = v$;*
   (b) *(Additive inverse) For each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = \mathscr{O}$;*
   (c) *(Associativity of addition) For all $v_1, v_2, v_3 \in V$,*
$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3);$$
(2) *(Abelian property)*
   (a) *(Commutativity of addition) For all $v_1, v_2 \in V$,*
$$v_1 + v_2 = v_2 + v_1;$$
(3) *(Module conditions)*
   (a) *For all $\lambda \in K$ and all $v_1, v_2 \in V$,*
$$\lambda \cdot (v_1 + v_2) = (\lambda \cdot v_1) + (\lambda \cdot v_2);$$
   (b) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 + \lambda_2) \cdot v = (\lambda_1 \cdot v) + (\lambda_2 \cdot v);$$
   (c) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v);$$
   (d) *For all $v \in V$,*
$$1 \cdot v = v.$$

In the above, for all $\lambda \in K$ and all $v, v_1, v_2 \in V$ we have denoted $+(v_1, v_2)$ by $v_1 + v_2$ and $\cdot(\lambda, v)$ by $\lambda \cdot v$.

In addition, for brevity, we will often write λv for $\lambda \cdot v$.

EXAMPLE 6.2.2 (The vector space K^n). By definition,
$$K^n = \{(x_1, \ldots, x_n) \ : \ x_i \in K, \ 1 \le i \le n\}.$$
The map $+ : K^n \times K^n \to K^n$ is defined by the rule
$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$
for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in K^n$. The map $\cdot : K \times K^n \to K^n$ is defined by the rule
$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$
for all $\lambda \in K$ and $(x_1, \ldots, x_n) \in K^n$.

Exercise 6.2.3. *Show that $(K^n, +, \cdot)$, defined in the example above, is a K-vector space.*

Exercise 6.2.4 (Cancelation rule). *Let $(V, +, \cdot)$ be a K-vector space. Show that if we have $v_1, v_2, w \in V$, then*

$$v_1 + w = v_2 + w \iff v_1 = v_2.$$

Exercise 6.2.5 (Unique additive identity). *Let $(V, +, \cdot)$ be a K-vector space. Fix an element $\mathcal{O} \in V$ such that for all $v \in V$, we have $v + \mathcal{O} = v$. Show that if $w \in V$ satisfies $v' + w = v'$ for all $v' \in V$, then $w = \mathcal{O}$.*

Exercise 6.2.6 (Unique additive inverse). *Let $(V, +, \cdot)$ be a K-vector space. Let $v \in V$. Fix an element $-v \in V$ such that $v + (-v) = \mathcal{O}$. Suppose that there is $w \in V$ such that $v + w = \mathcal{O}$. Show that $w = -v$.*

Exercise 6.2.7. *Let $(V, +, \cdot)$ be a K-vector space. Show the following properties hold for all $v, v_1, v_2 \in V$ and all $\lambda, \lambda_1, \lambda_2 \in K$.*

(1) $0v = \mathcal{O}$.
(2) $\lambda\mathcal{O} = \mathcal{O}$.
(3) $(-\lambda)v = -(\lambda v) = \lambda(-v)$.
(4) *If* $\lambda v = \mathcal{O}$, *then either* $\lambda = 0$ *or* $v = \mathcal{O}$.
(5) *If* $\lambda v_1 = \lambda v_2$, *then either* $\lambda = 0$ *or* $v_1 = v_2$.
(6) *If* $\lambda_1 v = \lambda_2 v$, *then either* $\lambda_1 = \lambda_2$ *or* $v = \mathcal{O}$.
(7) $-(v_1 + v_2) = (-v_1) + (-v_2)$.
(8) $v + v = 2v$, $v + v + v = 3v$, *and in general* $\sum_{i=1}^{n} v = nv$.

Exercise 6.2.8. *Consider the set of maps from a set S to K. Let us denote this set by* $\mathrm{Map}(S, K)$. *Define addition and multiplication maps*

$$+ : \mathrm{Map}(S, K) \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

and

$$\cdot : K \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

in the following way. For all $f, g \in \mathrm{Map}(S, K)$, set $f + g$ to be the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$. For all $\lambda \in K$ and all $f \in \mathrm{Map}(S, K)$, set $\lambda \cdot f$ to be the function defined by $(\lambda \cdot f)(x) = \lambda f(x)$ for all $x \in S$. Show that if $S \neq \emptyset$ then $(\mathrm{Map}(S, K), +, \cdot)$ is a K-vector space.

3. Sub-vector spaces

Definition 6.3.9 (sub-K-vector space)**.** *Let $(V, +, \cdot)$ be a K-vector space. A **sub-K-vector space** of $(V, +, \cdot)$ is a K-vector space $(V', +', \cdot')$ such that $V' \subseteq V$ and such that for all $v', v'_1, v'_2 \in V'$ and all $\lambda \in K$,*

$$v'_1 +' v'_2 = v'_1 + v'_2 \quad \text{and} \quad \lambda \cdot' v' = \lambda \cdot v'.$$

We will write $(V', +', \cdot') \subseteq (V, +, \cdot)$.

Definition 6.3.10. *If $(V, +, \cdot)$ is a K-vector space, and $V' \subseteq V$ is a subset, we say that V' is **closed under** $+$ **(resp. closed under** $\cdot$**)** if for all $v'_1, v'_2 \in V'$ (resp. for all $\lambda \in K$ and all $v' \in V'$) we have $v'_1 + v'_2 \in V'$ (resp. $\lambda \cdot v' \in V'$). In this case, we define*

$$+|_{V'} : V' \times V' \to V'$$

(resp. $\cdot|_{V'} : K \times V' \to V'$) to be the map given by $v'_1 +|_{V'} v'_2 = v'_1 + v'_2$ (resp. $\lambda \cdot |_{V'} v' = \lambda \cdot v'$), for all $v'_1, v'_2 \in V'$ (resp. for all $\lambda \in K$ and all $v' \in V'$).

REMARK 6.3.11. Note that if $(V', +', \cdot')$ is a sub-K-vector space of $(V, +, \cdot)$, then V' is closed under $+$ and $\cdot$.

Exercise 6.3.12. *Show that if a non-empty subset $V' \subseteq V$ is closed under $+$ and $\cdot$, then $(V', +|_{V'}, \cdot|_{V'})$ is a sub-K-vector space of $(V, +, \cdot)$.*

In other words, in the end, we tend to view a sub-K-vector space

$$(V', +', \cdot') \subseteq (V, +, \cdot)$$

as a subset $V' \subseteq V$ that is closed under $+$ and $\cdot$.

Exercise 6.3.13. *Show that if $(V', +', \cdot')$ is a sub-K-vector space of a K-vector space $(V, +, \cdot)$, then the additive identity element $\mathscr{O}' \in V'$ is equal to the additive identity element $\mathscr{O} \in V$.*

Exercise 6.3.14. *Recall the $\mathbb{R}$-vector space $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$ from Exercise 6.2.8. In this exercise, show that the subsets of $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ listed below are closed under $+$ and $\cdot$, and so define sub-$\mathbb{R}$-vector spaces of $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$.*

1. *The set of all polynomial functions.*
2. *The set of all polynomial functions of degree less than n.*
3. *The set of all functions that are continuos on an interval $(a, b) \subseteq \mathbb{R}$.*
4. *The set of all functions differentiable at a point $a \in \mathbb{R}$.*
5. *The set of all functions differentiable on an interval $(a, b) \subseteq \mathbb{R}$.*
6. *The set of all functions with $f(1) = 0$.*
7. *The set of all solutions to the differential equation $f'' + af' + bf = 0$ for some $a, b \in \mathbb{R}$.*

Exercise 6.3.15. *In this exercise, show that the subsets of $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ listed below are NOT closed under $+$ and $\cdot$, and so do not define sub-$\mathbb{R}$-vector spaces of $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$.*

1. *Fix $a \in \mathbb{R}$ with $a \neq 0$. The set of all functions with $f(1) = a$.*
2. *The set of all solutions to the differential equation $f'' + af' + bf = c$ for some $a, b, c \in \mathbb{R}$ with $c \neq 0$.*

4. Linear maps

Definition 6.4.16 (Linear map)**.** *Let $(V, +, \cdot)$ and $(V', +', \cdot')$ be K-vector spaces. A **linear map** $F : (V, +, \cdot) \to (V', +', \cdot')$ is a map of sets*

$$f : V \to V'$$

such that for all $\lambda \in K$ and $v, v_1, v_2 \in V$,

$$f(v_1 + v_2) = f(v_1) +' f(v_2) \quad \text{and} \quad f(\lambda \cdot v) = \lambda \cdot' f(v).$$

Note that we will frequently use the same letter for the linear map and the map of sets. The K-vector space $(V, +, \cdot)$ is called the **source** (or domain) of the linear map and the K-vector space $(V', +', \cdot')$ is called the **target** (or codomain) of the linear map. The set $f(V) \subseteq V'$ is called the **image** (or range) of f.

Exercise 6.4.17. *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. Show that the image of f is closed under $+', \cdot'$, and so defines a sub-K-vector space of the target $(V', +', \cdot')$.*

Exercise 6.4.18. *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. Show that $f(\mathscr{O}) = \mathscr{O}'$.*

Exercise 6.4.19. *Show that the following maps of sets define linear maps of the K-vector spaces.*

(1) *Let $(V,+,\cdot)$ be a K-vector space. Show that the identity map $f : V \to V$, given by $f(v) = v$ for all $v \in V$, is a linear map. This linear map will frequently be denoted by Id_V.*
(2) *Let $(V,+,\cdot)$ and $(V',+',\cdot')$ be K-vector spaces. Show that the zero map $f : V \to V'$, given by $f(v) = \mathcal{O}'$ for all $v \in V$, is a linear map.*
(3) *Let $(V,+,\cdot)$ be a K-vector space and let $\alpha \in K$. Show that the multiplication map $f : V \to V$ given by $f(v) = \alpha \cdot v$ for all $v \in V$ is a linear map. This linear map will frequently be denoted by $\alpha\,\mathrm{Id}_V$.*
(4) *Let $a_{ij} \in K$ for $1 \le i \le m$ and $1 \le j \le n$. Show that the map $f : K^n \to K^m$ given by*

$$f(x_1,\dots,x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j,\dots,\sum_{j=1}^{n} a_{ij}x_j,\dots,\sum_{j=1}^{n} a_{mj}x_j\right)$$

is a linear map.
(5) *Let $(V,+,\cdot)$ be the $\mathbb{R}$-vector space of all differentiable real functions $g : \mathbb{R} \to \mathbb{R}$. Let $(V',+',\cdot')$ be the $\mathbb{R}$-vector space of all real functions $g : \mathbb{R} \to \mathbb{R}$. Show that the map $f : (V,+,\cdot) \to (V',+',\cdot')$ that sends a differentiable function g to its derivative g' is a linear map.*
(6) *Let $(V,+,\cdot)$ be the $\mathbb{R}$-vector space of all continuous real functions $g : \mathbb{R} \to \mathbb{R}$. Show that the map $f : (V,+,\cdot) \to (V,+,\cdot)$ that sends a function $g \in V$ to the function $f(g) \in V$ determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \text{for all } x \in \mathbb{R}$$

is a linear map. Make sure to show that $f(g) \in V$ for all $g \in V$.

Definition 6.4.20 (Kernel)**.** *Let $f : (V,+,\cdot) \to (V',+',\cdot')$ be a linear map of K-vector spaces. The **kernel of** f (or Null space of f), denoted $\ker(f)$ (or $\mathrm{Null}(f)$), is the set*

$$\ker(f) := f^{-1}(\mathcal{O}') = \{v \in V : f(v) = \mathcal{O}'\}.$$

Exercise 6.4.21. *Let $f : (V,+,\cdot) \to (V',+',\cdot')$ be a linear map of K-vector spaces. Show that $\ker(f)$ is a sub-K-vector space of $(V,+,\cdot)$.*

Exercise 6.4.22. *Find the kernel of each of the linear maps listed below (see Problem 6.4.19).*

(1) *The linear map Id_V.*
(2) *The zero map $V \to V'$.*
(3) *The linear map $\alpha\,\mathrm{Id}_V$.*
(4) *Let $a_{ij} \in K$ for $1 \le i \le m$ and $1 \le j \le n$. The linear map $f : K^n \to K^m$ defined by*

$$f(x_1,\dots,x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j,\dots,\sum_{j=1}^{n} a_{ij}x_j,\dots,\sum_{j=1}^{n} a_{mj}x_j\right).$$

(5) *Let $(V,+,\cdot)$ be the $\mathbb{R}$-vector space of all differentiable real functions $g : \mathbb{R} \to \mathbb{R}$. Let $(V',+',\cdot')$ be the $\mathbb{R}$-vector space of all real functions $g : \mathbb{R} \to \mathbb{R}$. The linear map $f : (V,+,\cdot) \to (V',+',\cdot')$ that sends a differentiable function g to its derivative g'.*

(6) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all continous real functions $g : \mathbb{R} \to \mathbb{R}$. Let $a \in \mathbb{R}$. The linear map $f : (V, +, \cdot) \to (V, +, \cdot)$ that sends a function $g \in V$ to the function $f(g) \in V$ determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \text{for all } x \in \mathbb{R}.$$

Exercise 6.4.23. *Show that the composition of linear maps is a linear map.*

Definition 6.4.24 (Isomorphism). *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. We say that f is an isomorphism of K-vector spaces if there is a linear map $g : (V', +', \cdot') \to (V, +, \cdot)$ of K-vector spaces such that*

$$g \circ f = \mathrm{Id}_V \quad \text{and} \quad f \circ g = \mathrm{Id}_{V'}\,.$$

Exercise 6.4.25. *Show that a linear map is an isomorphism if and only if it is bijective.*

5. Bases and dimension

5.1. Linear maps determined by elements of a vector space. The basic example we are interested in is the following. Let V be a K-vector space. We fix

$$\mathbf{v} = (v_1, \ldots, v_n) \in V^n.$$

From this we obtain a map

$$L_{\mathbf{v}} : K^n \to V$$
$$(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i v_i.$$

Exercise 6.5.26. *Show that $L_{\mathbf{v}}$ is a linear map.*

5.2. Span, linear independence, and bases. For every permutation $\sigma \in \Sigma_n$, the symmetric group on n-letters, we set

$$\mathbf{v}^\sigma := (v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

Definition 6.5.27. *Let V be a K-vector space, and let $v_1, \ldots, v_n \in V$. Set $\mathbf{v} = (v_1, \ldots, v_n)$. We say:*

(1) *The elements $v_1, \ldots, v_n$ **span** V (or generate V) if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is surjective.*
(2) *The elements $v_1, \ldots, v_n$ are **linearly independent** if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is injective.*
(3) *The elements $v_1, \ldots, v_n$ are a **basis for** V if for every $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is an isomorphism.*

Exercise 6.5.28. *Let V be a K-vector space, and let $v_1, \ldots, v_n \in V$. Set $\mathbf{v} = (v_1, \ldots, v_n)$.*

(1) *The elements $v_1, \ldots, v_n$ **span** V (or generate V) if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is surjective.*
(2) *The elements $v_1, \ldots, v_n$ are **linearly independent** if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is injective.*
(3) *The elements $v_1, \ldots, v_n$ are a **basis for** V if for any $\sigma \in \Sigma_n$, the linear map $L_{\mathbf{v}^\sigma}$ is an isomorphism.*

Exercise 6.5.29. *Let V be a K-vector space, and let $v_1, \ldots, v_n \in V$.*

(1) *The elements $v_1, \ldots, v_n$ **span** V (or generate V) if for any $v \in V$, there exists $(a_1, \ldots, a_n) \in K^n$ such that $\sum_{i=1}^n a_i v_i = v$.*

(2) *The elements* $v_1, \ldots, v_n$ *are* ***linearly independent*** *if whenever* $(a_1, \ldots, a_n) \in K^n$ *and* $\sum_{i=1}^n a_i v_i = 0$, *we have* $(a_1, \ldots, a_n) = 0$.
(3) *The elements* $v_1, \ldots, v_n$ *are a* ***basis for*** V *if they span* V *and are linearly independent.*

5.3. Dimension. We start with the following motivational exercise:

Exercise 6.5.30. *If* $K^n \cong K^m$, *then* $n = m$.

Definition 6.5.31. *A* K*-vector space* V *is said to be of dimension* n *if there is an isomorphism* $V \cong K^n$.

Exercise 6.5.32. *Show that a* K*-vector space* V *has dimension* n *if and only if it has a basis consisting of* n *elements.*

6. Direct products of vector spaces

EXAMPLE 6.6.33. Suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are K-vector spaces. There is a K-vector space

$$(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$$

where $V_1 \times V_2$ is the product of the sets V_1 and V_2, where

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \to V_1 \times V_2$$

is defined by

$$(v_1, v_2) + (v_1', v_2') = (v_1 +_1 v_1', v_2 +_2 v_2')$$

and

$$\cdot : K \times (V_1 \times V_2) \to V_1 \times V_2$$

is defined by

$$\lambda \cdot (v_1, v_2) = (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2).$$

Exercise 6.6.34. *Show that the triple* $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$ *in the example above is a* K*-vector space.*

Definition 6.6.35 (Direct product)**.** *Suppose that* $(V_1, +_1, \cdot_1)$ *and* $(V_2, +_2, \cdot_2)$ *are* K*-vector spaces. We define the* direct product *of* $(V_1, +_1, \cdot_1)$ *and* $(V_2, +_2, \cdot_2)$, *written* $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2)$, *to be the* K*-vector space* $(V_1 \times V_2, +, \cdot)$ *defined above.*

Exercise 6.6.36. *Let* V_1 *and* V_2 *be* K*-vector spaces. Show the following:*

(1) *There is an injective linear map* $i_1 : V_1 \to V_1 \times V_2$ *given by* $v_1 \mapsto (v_1, \mathscr{O}_{V_2})$, *and a surjective linear map* $p_1 : V_1 \times V_2 \to V_1$ *given by* $(v_1, v_2) \mapsto v_1$.
(2) *There is an injective linear map* $i_2 : V_1 \to V_1 \times V_2$ *given by* $v_2 \mapsto (\mathscr{O}_{V_1}, v_2)$, *and a surjective linear map* $p_2 : V_1 \times V_2 \to V_2$ *given by* $(v_1, v_2) \mapsto v_2$.

7. Quotient vector spaces

Suppose that $(V, +, \cdot)$ is a K-vector space, and $W \subseteq V$ is a sub-K-vector space. Define an equivalence relation on V by the rule

$$v_1 \sim v_2 \iff v_1 - v_2 \in W.$$

Exercise 6.7.37. *Show that this defines an equivalence relation on* V.

Let V/W be the set of equivalence classes, and let

$$\pi : V \longrightarrow V/W$$

be the quotient map of sets. For any element $\nu \in V/W$, there is an element $v \in V$ such that $\nu = [v]$, where $[v]$ is the equivalence class of v.

Exercise 6.7.38. *Let V be a K-vector space and suppose that $W \subseteq V$ is a sub-K-vector space.*

(1) *Suppose that $[v_1], [v_2] \in V/W$. Show that the rule*

$$[v_1] + [v_2] = [v_1 + v_2]$$

defines a map

$$+ : V/W \times V/W \to V/W.$$

(2) *Suppose that $\lambda \in K$ and $[v] \in V/W$. Show that the rule*

$$\lambda \cdot [v] = [\lambda \cdot v]$$

defines a map

$$\cdot : K \times V/W \to V/W.$$

(3) *Show that V/W is a K-vector space with $+$ and $\cdot$ defined as above.*
(4) *Show that $\pi : V \to V/W$ is a surjective linear map with kernel W.*

Definition 6.7.39 (Quotient K-vector space)**.** *Let V be a K-vector space and let $W \subseteq V$ be a sub-K-vector space. The quotient (K-vector space) of V by W is the K-vector space V/W constructed above.*

Exercise 6.7.40. *Suppose that $\phi : V \twoheadrightarrow V'$ is a surjective linear map of K-vector spaces.*

(1) *Show that $V' \cong V/\ker\phi$.*
(2) *If V' is finite dimensional, show that $V \cong (\ker\phi) \times V'$.*
(3) *If V and V' are finite dimensional, show that $\dim V = \dim V' + \dim(\ker\phi)$.*

8. Further exercises

Exercise 6.8.41. *Find an example of a triple $(V, +, \cdot)$ satisfying all of the conditions of the definition of a K-vector space, except for condition (3)(d).*

Exercise 6.8.42. *Suppose that $L : K^n \to K^m$ is a linear map. For $j = 1, \dots, n$ define $e_j = (0, \dots, 1, \dots, 0) \in K^n$ to be the element with all entries 0 except for the j-th place, which is 1. Similarly, for $i = 1, \dots, m$ define $f_i^\vee : K^m \to K$ to be the linear map defined by $(y_1, \dots, y_m) \mapsto y_i$. Show that L is the same as the linear map defined in Example 6.4.19(4) with the matrix $A \in \mathrm{M}_{m\times n}(K)$ defined by $A_{ij} = a_{ij} = f_i^\vee(L(e_j))$.*