Exercise 9.33

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 9.33 from Fraleigh [Fra03, §9]:

Exercise 9.33. Consider S_n for a fixed $n \geq 2$, and let σ be a fixed odd permutation. Show that every odd permutation in S_n is a product of σ and some permutation in A_n.

Solution. Let σ' be an odd permutation in S_n. We must show that there exists an even permutation $\mu \in A_n$ such that $\sigma' = \sigma\mu$. Indeed, we may take $\mu = \sigma^{-1}\sigma'$, since, as the product of two odd permutations, it is an even permutation (see below), and $\sigma' = \sigma(\sigma^{-1}\sigma') = \sigma\mu$. □

For completeness, let's prove directly the assertion above that $\sigma^{-1}\sigma'$ is even. From the definition of an odd permutation, there exist a finite number of transpositions $\tau_1, \dots, \tau_m$ for some odd $m \in \mathbb{N}$ such that

$$\sigma = \tau_1 \dots \tau_m.$$

Similarly, since σ' is also an odd permutation, there exist a finite number of transpositions $\tau'_1, \dots, \tau'_\ell$ for some odd $\ell \in \mathbb{N}$ such that $\sigma' = \tau'_1 \dots \tau'_\ell$. Consider now the permutation

$$\mu = \sigma^{-1}\sigma'.$$

I claim that this lies in A_n. Indeed we have

$$\mu = \sigma^{-1}\sigma' = \underbrace{\tau_m \dots \tau_1 \tau'_1 \dots \tau'_\ell}_{m+\ell}.$$

The sum of two odd numbers is even, and so it follows that this is an even permutation.

Date: October 5, 2021.

REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

UNIVERSITY OF COLORADO, DEPARTMENT OF MATHEMATICS, CAMPUS BOX 395, BOULDER, CO 80309

Email address: casa@math.colorado.edu