Exercise 4.4

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 4.4 from Fraleigh [Fra03, §4]:

Exercise 4.4. Let $*$ be defined on $\mathbb{Q}$ by letting $a * b = ab$. Determine whether the binary structure $\langle \mathbb{Q}, * \rangle$ is a group. If it is not a group, give the first condition (or "group axiom") $\mathscr{G}_1$, $\mathscr{G}_2$, or $\mathscr{G}_3$, from [Fra03, Definition 4.1] that does not hold.

Solution. The binary structure $\langle \mathbb{Q}, * \rangle$ is not a group; the first condition (or "group axiom") that does not hold is $\mathscr{G}_3$. Indeed while $*$ is associative, i.e., $\mathscr{G}_1$ holds (multiplication of rational numbers is associative), and $1 \in \mathbb{Q}$ is an identity element for the binary structure $\langle \mathbb{Q}, * \rangle$, i.e., $\mathscr{G}_2$ holds (for all $a \in \mathbb{Q}$ we have $1 * a = a * 1 = a$), the element $0 \in \mathbb{Q}$ does not have an inverse, i.e., $\mathscr{G}_3$ fails (there is no element $a \in \mathbb{Q}$ such that $a * 0 = 0 * a = 1$). □

Remark 0.1. Note that letting $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ be the non-zero rational numbers, and letting $*$ be defined on $\mathbb{Q}^*$ by letting $a * b = ab$, we have that $\langle \mathbb{Q}^*, * \rangle$ *is* a group.

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References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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