# Exercise 31.31

## Abstract Algebra 1
## MATH 3140

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ABSTRACT. This is Exercise 31.31 from Fraleigh [Fra03, §31]:

**Exercise 31.31.** Show that if $F$, $E$, and $K$ are fields with $F \leq E \leq K$, then $K$ is algebraic over $F$ if and only if $E$ is algebraic over $F$, and $K$ is algebraic over $E$. (You must *not* assume the extensions are finite.)

*Solution.* ( $\implies$ ) First assume that $K$ is algebraic over $F$. We will start by showing that $K$ is algebraic over $E$. To this end, let $\alpha \in K$. We know that $\alpha$ is algebraic over $F$, so it satisfies a monic polynomial $f(x) \in F[x]$. But since $F[x] \subseteq E[x]$, we see that $\alpha$ satisfies a monic polynomial over $E[x]$, and so $\alpha$ is algebraic over $E$. Next let us show that $E$ is algebraic over $F$. To this end, since $E \subseteq K$, and by definition, every element of $K$ is algebraic over $F$, we have that every element of $E$ is algebraic over $F$, and we are done.

( $\impliedby$ ) Next let us assume that $K$ is algebraic over $E$, and $E$ is algebraic over $F$, and let us show that $K$ is algebraic over $F$. So let $\alpha \in K$. We know that $K$ is algebraic over $E$, so that $\alpha$ satisfies a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in E[x]. \tag{0.1}$$

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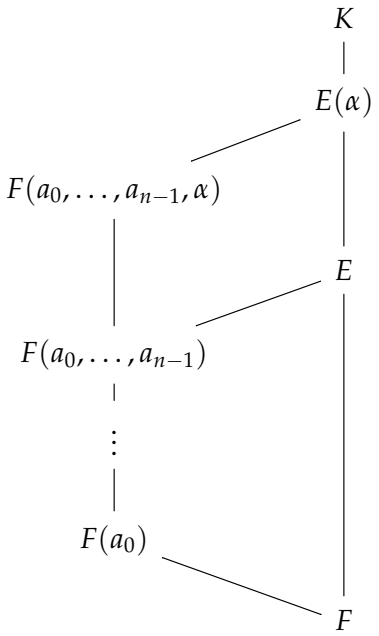
Then we consider the tower of field extensions




Since $E$ is algebraic over $F$, the extensions $F \leq F(a_0) \leq \cdots \leq F(a_0,\ldots,a_{n-1})$ are algebraic (for instance, this follows from the implication ( $\Longrightarrow$ ) of this problem, which we proved in the first paragraph). By [Fra03, Theorem 30.23], every simple algebraic extension is finite, and by [Fra03, Corollary 31.6], compositions of finite extensions are finite, and so in conclusion, $F(a_0,\ldots,a_{n-1})$ is finite over $F$ (you can also just use [Fra03, Theorem 31.11]).

Finally, the fact that $\alpha$ satisfies the monic polynomial (0.1) implies that $\alpha$ is algebraic over $F(a_1,\ldots,a_{n-1})$. In other words, the extension $F(a_0,\ldots,a_{n-1}) \leq F(a_0,\ldots,a_{n-1},\alpha)$ is a simple algebraic extension, and is therefore finite. In summary, the extension $F \leq F(a_0,\ldots,a_{n-1},\alpha)$ is finite [Fra03, Corollary 31.6], and is therefore algebraic by [Fra03, Theorem 31.3]. Thus $\alpha$ is algebraic over $F$, and we are done. □

## REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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