Exercise 21.2

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 21.2 from Fraleigh [Fra03, §21]:

Exercise 21.2. Describe the field F of quotients of the integral subdomain

$$D = \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$$

of $\mathbb{R}$.

More precisely, following the notation of [Fra03, Theorem 21.6], describe the injective ("one-to-one") homomorphism of rings $\psi : F \to \mathbb{R}$ that gives an isomorphism of F with a subfield K of $\mathbb{R}$ such that $\psi(a) = a$ for all $a \in D$, and give a concise description of K.

Solution. By the construction of the field of quotients F, the map $\psi : F \to \mathbb{R}$ from [Fra03, Theorem 21.6] is given by the rule

$$\psi\left(n + m\sqrt{2}, n' + m'\sqrt{2}\right) = \frac{n + m\sqrt{2}}{n' + m'\sqrt{2}}.$$

This gives an isomorphism of F with the subfield

$$K := \{n + m\sqrt{2} : n, m \in \mathbb{Q}\} \subseteq \mathbb{R}.$$

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Indeed, it suffices to show that $\psi[F] = K$. First let $\frac{a}{b} + \frac{c}{d}\sqrt{2} \in K$. Then, since $\frac{a}{b} + \frac{c}{d}\sqrt{2} = \frac{ad}{bd} + \frac{bc}{bd}\sqrt{2}$, we have that $\psi(ad + bc\sqrt{2}, bd) = \frac{a}{b} + \frac{c}{d}\sqrt{2}$. Thus $K \subseteq \psi[F]$. On the other hand,

$$\begin{aligned}
\psi\left(n + m\sqrt{2}, n' + m'\sqrt{2}\right) &= \frac{n + m\sqrt{2}}{n' + m'\sqrt{2}} \\
&= \frac{n + m\sqrt{2}}{n' + m'\sqrt{2}} \cdot \frac{n' - m'\sqrt{2}}{n' - m'\sqrt{2}} \\
&= \frac{(nn' - 2mm') + (mn' - nm')\sqrt{2}}{n'^2 - 2m'^2} \\
&= \frac{nn' - 2mm'}{n'^2 - 2m'^2} + \frac{mn' - m'n}{n'^2 - 2m'^2}\sqrt{2}
\end{aligned}$$

Thus $\psi[F] \subseteq K$, and we have that $\psi[F] = K$, completing the proof. □

References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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