Exercise 18.42

Abstract Algebra 1
MATH 3140

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ABSTRACT. This is Exercise 18.42 from Fraleigh [Fra03, §18]:

Exercise 18.42. Show that the unity element in a subfield of a field must be the unity element of the whole field (in contrast to Exercise 18.32 for rings in general).

A field is an integral domain, and since the statement holds for integral domains as well, and our proof extends easily, we will prove this more general fact:

> *Let D' be an integral domain, and let $D \subseteq D'$ be a subring that is also an integral domain. Let 1_D to be the unity element of D and let $1'_D$ be the unity element of D'. Then $1_D = 1_{D'}$.*

Proof. We recall now for later reference that $0_D = 0_{D'}$ (since the identity element of a subgroup is the identity element of the group).

Now let $a \in D$ be any non-zero element (such an element exists since $1_D \neq 0_D$ by the definition of an integral domain). We have in D that

$$a \cdot 1_D = a.$$

Multiplication in D is induced by that of D', and so this equality also holds in D'. Similarly, in D' we have

$$a \cdot 1_{D'} = a$$

so that

$$a \cdot 1_D = a \cdot 1_{D'}. \tag{0.1}$$

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If D' were a field, then we could multiply by $a^{-1} \in D'$ (the multiplicative inverse of a in D')[1] and obtain that $1_D = 1'_D$. On the other hand, if D' is not a field, then a may not have a multiplicative inverse in D', and this argument would not work.

However, whether or not D' is a field, (0.1) still implies that

$$a \cdot (1_D - 1_{D'}) = 0_{D'}.$$

Then, since we are assuming D' is an integral domain, and by assumption $a \neq 0_D (= 0_{D'})$, it follows that

$$1_D - 1_{D'} = 0_{D'}.$$

Thus we have shown $1_D = 1_{D'}$. □

In fact, we can even prove the following more general statement:

Let R' be a ring with no zero divisors, and let $R \subseteq R'$ be a subring with unity $1_R \neq 0_R$. Then 1_R is a unity element for R'.

For an even more general statement, you can see Exercise 26.21.

Solution. Let R' be a ring with no zero divisors, and let $R \subseteq R'$ be a subring with unity $1_R \neq 0_R$. We will show 1_R is a unity element for R'.

We want to show that for all $r' \in R'$, we have

$$1_R \cdot r' = r'. \tag{0.2}$$

This is equivalent to showing

$$1_R \cdot r' - r' = 0, \tag{0.3}$$

which, since $1_R \neq 0'$ and R' has no zero divisors, is equivalent to showing

$$1_R(1_R \cdot r' - r') = 0. \tag{0.4}$$

[1]Note that for this argument, we need to use that a has a multiplicative inverse in D' (even if D and D' are fields, we do not yet know that the multiplicative inverse of a in D agrees with the multiplicative inverse of a in D'; we have not yet proven that). In fact, the example $\mathbb{Z}_2 \times \{0\} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$ shows that it is not enough to only assume that D is a field.

In other words, to show that 1_R is unity for R', it suffices to show that (0.4) holds for all $r' \in R$. We now prove this:

$$\begin{aligned} 1_R(1_R \cdot r' - r') &= 1_R \cdot 1_R \cdot r' - 1_R r' \\ &= 1_R \cdot r' - 1_R r' \\ &= 0. \end{aligned}$$

This completes the proof. □

References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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