Exercise 13.47

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 13.47 from Fraleigh [Fra03, §13]:

Exercise 13.47. Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or injective (a "one-to-one") map.

Solution. Let $\phi : G \to G'$ be a group homomorphism where $|G|$ is a prime. Let $e' \in G'$ be the identity element. The problem asks us to show that $\phi(g) = e'$ for all $g \in G$, or that ϕ is injective ("one-to-one").

To prove this, let us consider $\ker \phi$. The kernel of a homomorphism is a subgroup of G, and, since $|G|$ is finite, $|\ker \phi|$ divides $|G|$ (Theorem of Lagrange [Fra03, p.100]). By virtue of the fact that $|G|$ is prime, it follows that either $|\ker \phi| = 1$ or $|\ker \phi| = |G|$. That is, either $\ker \phi = \{e\}$, where e is the identity element of G, or $\ker \phi = G$.

In the former case (i.e., $\ker \phi = \{e\}$), ϕ is one-to-one (a homomorphism is injective ("one-to-one") if and only if the kernel is trivial [Fra03, Corollary 13.18, p.131]). In the latter case, $\phi(g) = e'$ for all $g \in G$, from the definition of the kernel. $\square$

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References

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

Email address: casa@math.colorado.edu