**Exercise 0.18**

**Abstract Algebra 1**
**MATH 3140**

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ABSTRACT. This is Exercise 0.18 from Fraleigh [Fra03, §0]:

**Exercise 0.18.** For any set $A$, finite or infinite, let $B^A$ be the set of all functions mapping $A$ into the set $B = \{0,1\}$ (maps from $A$ to $B = \{0,1\}$). Show that the cardinality of $B^A$ is the same as the cardinality of the set $\mathscr{P}(A)$. [Hint: Each element of $B^A$ determines a subset of $A$ in a natural way.]

*Solution.* To show that the cardinality of $B^A$ is the same as the cardinality of the set $\mathscr{P}(A)$, we need to construct a bijective map (one-to-one and onto function)

$$\phi : B^A \longrightarrow \mathscr{P}(A).$$

We define $\phi$ as follows. Given a map (function) $f : A \to \{0,1\}$, we define

$$\phi(f) := \{a \in A : f(a) = 1\} \subseteq A.$$

Now we must show it is bijective (one-to-one and onto). First let us show it is injective (one-to-one). So suppose that $f, g \in B^A$, and $\phi(f) = \phi(g)$. In other words,

$$\phi(f) = \{a \in A : f(a) = 1\} = \phi(g) = \{a \in A : g(a) = 1\}.$$

Since any map (function) $A \to \{0,1\}$ is determined by the elements of $a$ that it sends to 1 (it must send the remaining elements to 0), we see that $f = g$. Thus $\phi$ is injective (one-to-one).

Now let us show that $\phi$ is surjective (onto). Let $S \subseteq A$. Then let $1_S : A \to \{0,1\}$ be the map (function) defined by

$$1_S(a) = \begin{cases} 1, & a \in S \\ 0, & a \notin S. \end{cases}$$

Then $\phi(1_S) = S$, and therefore $\phi$ is surjective (onto). □

*Date*: September 3, 2021.

*Remark* 0.1. As an alternate approach, to show that $\phi$ is bijective (one-to-one and onto), it suffices to construct an inverse map (function)

$$\psi : \mathscr{P}(A) \longrightarrow B^A;$$

i.e., a map (function) $\psi$ as above such that for all $f \in B^A$ we have $\psi(\phi(f)) = f$, and for all $S \subseteq A$ we have $\phi(\psi(S)) = S$. We construct $\psi$ as follows. Given a subset $S \subseteq A$, we define $\psi(S) : A \to \{0,1\}$ by the rule

$$\psi(S)(a) = \begin{cases} 1, & a \in S \\ 0, & a \notin S. \end{cases}$$

You can check that $\psi$ is an inverse map (function) for $\phi$. (You can also see that the map (function) $1_S$ in the solution above is equal to the map (function) $\psi(S)$.)

## REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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