Midterm 2

Linear Algebra

MATH 2130

Spring 2021

Friday March 19, 2021

NAME: **Enter your name here** ______________________________

PRACTICE EXAM

SOLUTIONS

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

- This exam is closed book.
- You may use only paper and pencil.
- You may not use any other resources whatsoever.
- You will be graded on the clarity of your exposition.

1. (10 points) • Let $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, and suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are K-vector spaces. Recall that there is set $V_1 \times V_2$, called the product of V_1 and V_2, whose elements consist of the ordered pairs (v_1, v_2) such that $v_1 \in V_1$ and $v_2 \in V_2$.

Define a map of sets

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \to V_1 \times V_2$$

$$(v_1, v_2) + (v_1', v_2') = (v_1 +_1 v_1', v_2 +_2 v_2')$$

and a map of sets

$$\cdot : K \times (V_1 \times V_2) \to V_1 \times V_2$$

$$\lambda \cdot (v_1, v_2) = (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2).$$

Show that the triple $(V_1, +_1, \cdot_1) \times (V_2, +_2, \cdot_2) := (V_1 \times V_2, +, \cdot)$ is a K-vector space. We call this the product of the vector spaces $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$.

SOLUTION

Solution. For brevity, set $V = V_1 \times V_2$. We must check that $(V, +, \cdot)$ satisfies the conditions of being a K-vector space.

1. (Group laws)

   (a) (Additive identity) I claim there exists an element $\mathscr{O} \in V$ such that for all $v \in V$, $v + \mathscr{O} = v$.

   Indeed, set $\mathscr{O} = (\mathscr{O}_1, \mathscr{O}_2)$, where $\mathscr{O}_1 \in V_1$ is the additive identity for $(V_1, +_1, \cdot_1)$ and $\mathscr{O}_2 \in V_2$ is the additive identity for $(V_2, +_2, \cdot_2)$. Then for any $v = (v_1, v_2) \in V = V_2 \times V_2$, we have

$$\begin{aligned} v + \mathscr{O} &= (v_1, v_2) + (\mathscr{O}_1, \mathscr{O}_2) \\ &= (v_1 +_1 \mathscr{O}_1, v_2 +_2 \mathscr{O}_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\ &= (v_1, v_2) && \text{(1)(a) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\ &= v. \end{aligned}$$

   (b) (Additive inverse) I claim that for each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = \mathscr{O}$.

   Indeed, given $v = (v_1, v_2) \in V = V_1 \times V_2$, set $-v = (-v_1, -v_2)$, where $-v_1 \in V_1$ is the additive inverse of v_1, and $-v_2 \in V_2$ is the additive inverse of v_2. Then

$$\begin{aligned}
v + (-v) &= (v_1, v_2) + (-v_1, -v_2) \\
&= (v_1 +_1 (-v_1), v_2 +_2 (-v_2)) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= (\mathscr{O}_1, \mathscr{O}_2) && \text{(1)(b) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\
&= \mathscr{O}.
\end{aligned}$$

(c) (Associativity of addition) I claim that for all $u, v, w \in V$,

$$(u + v) + w = u + (v + w).$$

Indeed, given $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
(u + v) + w &= ((u_1, u_2) + (v_1, v_2)) + (w_1, w_2) \\
&= (u_1 +_1 v_1, u_2 +_2 v_2) + (w_1, w_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= ((u_1 +_1 v_1) +_1 w_1, (u_2 +_2 v_2) +_2 w_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= (u_1 +_1 (v_1 +_1 w_1), u_2 +_2 (v_2 +_2 w_2)) && \text{(1)(c) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\
&= (u_1, u_2) + (v_1 +_1 w_1, v_2 +_2 w_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= (u_1, u_2) + ((v_1, v_2) + (w_1, w_2)) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= u + (v + w).
\end{aligned}$$

2. (Abelian property)

(a) (Commutativity of addition) For all $u, v \in V$,

$$u + v = v + u.$$

Indeed, given $u = (u_1, u_2), v = (v_1, v_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
u + v &= (u_1, u_2) + (v_1, v_2) \\
&= (u_1 +_1 v_1, u_2 +_2 v_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= (v_1 +_1 u_1, v_2 +_2 u_2) && \text{(2)(a) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\
&= (v_1, v_2) + (u_1, u_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= v + u.
\end{aligned}$$

3. (Module conditions)

(a) I claim that for all $\lambda \in K$ and all $u, v \in V$,

$$\lambda \cdot (u + v) = (\lambda \cdot u) + (\lambda \cdot v).$$

Indeed, given $\lambda \in K$ and $u = (u_1, u_2), v = (v_1, v_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
\lambda \cdot (u + v) &= \lambda \cdot ((u_1, u_2) + (v_1, v_2)) \\
&= \lambda \cdot (u_1 +_1 v_1, u_2 + v_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= (\lambda \cdot_1 (u_1 +_1 v_1), \lambda \cdot_2 (u_2 +_2 v_2)) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= (\lambda \cdot_1 u_1 + \lambda \cdot_1 v_1, \lambda \cdot_2 u_2 + \lambda \cdot_2 v_2) && \text{(3)(a) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\
&= (\lambda \cdot_1 u_1, \lambda \cdot_2 u_2) + (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= \lambda \cdot (u_1, u_1) + \lambda \cdot (v_1, v_2) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= (\lambda \cdot u) + (\lambda \cdot v)
\end{aligned}$$

(b) I claim that for all $\lambda, \mu \in K$, and all $v \in V$,

$$(\lambda + \mu) \cdot v = (\lambda \cdot v) + (\mu \cdot v).$$

Indeed, given $\lambda, \mu \in K$ and $v = (v_1, v_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
(\lambda + \mu) \cdot v &= (\lambda + \mu) \cdot (v_1, v_2) \\
&= ((\lambda + \mu) \cdot_1 v_1, (\lambda + \mu) \cdot_2 v_2) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= (\lambda \cdot_1 v_1 + \mu \cdot_1 v_1, \lambda \cdot_2 v_2 + \mu \cdot_2 v_2) && \text{(3)(b) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\
&= (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2) + (\mu \cdot_1 v_1, \mu \cdot_2 v_2) && \text{Def. of } + \text{ in } (V, +, \cdot) \\
&= \lambda \cdot (v_1, v_2) + \mu \cdot (v_1, v_2) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= (\lambda \cdot v) + (\mu \cdot v).
\end{aligned}$$

(c) For all $\lambda, \mu \in K$, and all $v \in V$,

$$(\lambda\mu) \cdot v = \lambda \cdot (\mu \cdot v).$$

Indeed, given $\lambda, \mu \in K$ and $v = (v_1, v_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
(\lambda\mu) \cdot v &= (\lambda\mu) \cdot (v_1, v_2) \\
&= ((\lambda\mu) \cdot_1 v_1, (\lambda\mu) \cdot_2 v_2) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= (\lambda \cdot_1 (\mu \cdot_1 v_1), \lambda \cdot_2 (\mu \cdot_2 v_2)) && \text{(3)(c) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\
&= \lambda \cdot (\mu \cdot_1 v_1, \mu \cdot_2 v_2) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= \lambda \cdot (\mu \cdot (v_1, v_2)) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= \lambda \cdot (\mu \cdot v).
\end{aligned}$$

(d) I claim that for all $v \in V$,

$$1 \cdot v = v.$$

Indeed, given $v = (v_1, v_2) \in V = V_1 \times V_2$, we have

$$\begin{aligned}
1 \cdot v &= 1 \cdot (v_1, v_2) \\
&= (1 \cdot_1 v_1, 1 \cdot_2 v_2) && \text{Def. of } \cdot \text{ in } (V, +, \cdot) \\
&= (v_1, v_2) && \text{(3)(d) for } (V_1, +_1, \cdot_1) \text{ and } (V_2, +_2, \cdot_2) \\
&= v.
\end{aligned}$$

□

2. (10 points) • Find the determinant of each of the following matrices.

(a) $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(b) $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}$

SOLUTION

Solution to (a). We have $\boxed{\det A = -1}$ The fastest way to see this may be to expand off of the third column; however, to use the standard method, we have

$$\det A = (4)[(-2)(0) - (0)(1)] - (-1)[(-1)(0) - (0)(0)] + (1)[(-1)(1) - (-2)(0)] = -1.$$

□

Solution to (b). We have $\boxed{\det B = -2}$

We use row operations:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -2$$

□

3. (10 points) • Let $V = \mathbb{R}[x]$ be the real vector space of real polynomial functions. Let

$$D : V \to V$$

$$p(x) \mapsto p'(x)$$

be the derivative map; i.e., $D(p(x)) = p'(x)$ for all polynomials $p(x) \in V$. Let

$$E : V \to V$$

$$p(x) \mapsto \int_0^x p(t)\, dt$$

be the integration map that sends a polynomial $p(x) \in V$ to the polynomial $q(x) \in V$ given by the rule $q(x) = \int_0^x p(t)\, dt$. It is a fact (which you can use without proof) that D and E are linear maps.

(a) *Show that D is surjective, but not injective.*

(b) *Show that E is injective, but not surjective.*

SOLUTION

Solution to (a). We will show first that $(ED)(p) = p - p(0)$, and $(DE)(p) = p$ (we will only use the latter). We have

$$\begin{aligned}
((ED)(p))(x) &= (E(D(p)))(x) \\
&= \int_0^x D(p)(t)dt = \int_0^x p'(t)dt \\
&= p(x) - p(0) \\
((DE)(p))(x) &= (D(E(p)))(x) \\
&= \frac{d}{dx}(E(p)(x)) \\
&= \frac{d}{dx}\int_0^x p(t)dt \\
&= p(x).
\end{aligned}$$

Now we will apply this to show (a). Since $DE : V \to V$ is the identity, and in particular is surjective, we must have that D is surjective (more directly, we can prove the surjectivity of D by observing that every polynomial has an anti-derivative that is a polynomial). On the other hand, D is not injective, since $D(p) = 0$ for every constant polynomial p. □

Solution to (b). Since $DE : V \to V$ is the identity, and in particular is injective, we must have that E is injective (more directly, we can prove the injectivity of E by observing that every polynomial has an anti-derivative that is a polynomial of degree at least 1, and arguing from there). On the other hand, E is not surjective, since, for instance, there is no polynomial p such that $\int_0^x p(t)dt = 1$ for all x (for instance plug in $x = 0$ and we get $\int_0^0 p(t)dt = 0 \neq 1$). □

4. (10 points) • Suppose we have a two state Markov chain with stochastic matrix

$$P = \begin{pmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{pmatrix}$$

Given the probability vector $v = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$, *find* $\lim_{n\to\infty} P^n v$.

SOLUTION

Solution. The solution is $\boxed{\lim_{n\to\infty} P^n v = \begin{pmatrix} 5/14 \\ 9/14 \end{pmatrix}}$

Indeed, since P is a stochastic matrix with positive entries, given any probability vector v, we have $\lim_{n\to\infty} P^n v$ is the unique probability vector that is an eigenvector with eigenvalue 1. So to obtain the solution, we first find an eigenvector with eigenvalue 1. For this, we are trying to find the kernel of

$$1 \cdot I - P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.9 & -0.5 \\ -0.9 & 0.5 \end{pmatrix}$$

We put the matrix in reduced row echelon form:

$$\begin{pmatrix} 1 & -5/9 \\ 0 & 0 \end{pmatrix}$$

and then we modify the matrix:

$$\begin{pmatrix} 1 & -5/9 \\ 0 & -1 \end{pmatrix}$$

This tells us that $\begin{pmatrix} -5/9 \\ -1 \end{pmatrix}$, or more conveniently, $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$, gives a basis for the $\lambda = 1$ eigenspace. The corresponding probability vector is $\begin{pmatrix} 5/14 \\ 9/14 \end{pmatrix}$. □

5. (10 points) $\bullet$ Consider the following real matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

(a) *Find the characteristic polynomial $p_A(t)$ of A.*

(b) *Find the eigenvalues of A.*

(c) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

(d) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

SOLUTION

Solution to (a). We have

$$\begin{aligned} p_A(t) &= \begin{vmatrix} t-2 & 1 & -1 \\ 0 & t-3 & 1 \\ -2 & -1 & t-3 \end{vmatrix} \\ &= (t-2)[(t-3)^2 - (1)(-1)] - (1)[0-(1)(-2)] + (-1)[0-(t-3)(-2)] \\ &= (t-2)[t^2-6t+10] - 2 + \underbrace{(t-3)(-2)}_{-2t+6} \\ &= (t^3 - 6t^t + 10t - 2t^2 + 12t - 20) - 2 + (6-2t) \\ &= t^3 - 8t^2 + 20t - 16. \end{aligned}$$

In other words, the solution is:

$$\boxed{p_A(t) = t^3 - 8t^2 + 20t - 16.}$$

$\square$

As a quick partial check of the solution, observe that

$$\operatorname{tr}(A) = 8$$

$$\det A = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 2 & 2 \end{vmatrix} = 2(6+2) = 16.$$

confirming the computation of the coefficients of t^2 and t^0, since we know that

$$p_A(t) = t^3 - \operatorname{tr}(A)t^2 + \alpha t + (-1)^3 \det(A)$$

for some real number $\alpha \in \mathbb{R}$.

Solution to (b). One can easily check that

$$p_A(2) = 2^3 - 8 \cdot 2^2 + 20 \cdot 2 - 16 = 8 - 32 + 40 - 16 = 48 - 48 = 0.$$

Thus $(t-2)$ is a factor of $p_A(t)$, so that we have

$$p_A(t) = (t-2)(t^2 - 6t + 8) = (t-2)(t-2)(t-4).$$

Thus the eigenvalues are

$$\boxed{\lambda = 2, 4.}$$

□

Solution to (c). To find a basis for the $\lambda = 2$ eigenspace E_2, we compute

$$E_2 := \ker(2I - A) = \ker \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -1 & -1 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We add rows, and get the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ {\color{red}0} & {\color{red}0} & {\color{green}-1} \end{pmatrix}$$

Thus we have

$$E_2 = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ {\color{green}-1} \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Now we compute a basis for the $\lambda = 4$ eigenspace E_4. We have

$$E_4 = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ {\color{red}0} & {\color{red}0} & {\color{green}-1} \end{pmatrix}$$

Thus we have

$$E_4 = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \\ {\color{green}-1} \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Thus the solution to the problem is:

The eigenspaces for A are E_2 and E_4, and we have that $\begin{pmatrix} 1 \\ -1 \\ {\color{green}-1} \end{pmatrix}$ is a basis for E_2 and $\begin{pmatrix} -1 \\ 1 \\ {\color{green}-1} \end{pmatrix}$ is a basis for E_4.

□

Note that we can easily double check that the given basis elements are eigenvectors with the stated eigenvalues.

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+1-1 \\ -3+1 \\ 2-1-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-1-1 \\ 3+1 \\ -2+1-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$$

Solution to (d). No. A is not diagonalizable since we showed in part (c) that there does not exists a basis of $\mathbb{R}^3$ consisting of eigenvectors for A. □

6. (10 points) • Consider the following matrix:

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}$$

(a) *What is the sum of the roots of the characteristic polynomial of B?*

(b) *What is the product of the roots of the characteristic polynomial of B?*

(c) *Are all of the roots of the characteristic polynomial of B real?*

SOLUTION

Solution to (a). (a) The sum of the roots of the characteristic polynomial of B is equal to the trace of B. So we have

$$\operatorname{tr} B = 0 + 0 + 2 + 1 + 8 + 0 = 11.$$

So the sum of the roots is $\boxed{11}$. □

Solution to (b). The product of the roots of the characteristic polynomial of B is equal to the determinant of B. Since B is block-upper-triangular, we could compute the determinant that way; but the fourth and fifth rows are linearly dependent, so the determinant is 0. Thus the product of the roots is $\boxed{0}$. □

Solution to (c). $\boxed{\text{No,}}$ not all of the roots of the characteristic polynomial are real. We have

$$P_B(t) = \left|\begin{array}{cc|cccc} t & -1 & 0 & -2 & 1 & 0 \\ 1 & t & -2 & -1 & -1 & -1 \\ \hline 0 & 0 & t-2 & 0 & 2 & 0 \\ 0 & 0 & 0 & t-1 & -4 & -3 \\ 0 & 0 & 0 & -2 & t-8 & -6 \\ 0 & 0 & 0 & -3 & 3 & t \end{array}\right| = (t^2+1)p(t)$$

where $p(t)$ is the determinant of the lower right block in the matrix above. Thus $\pm i$ are roots of $p_B(t)$.

□

7. (10 points) • Consider the two dimensional discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{pmatrix} 1.7 & 0.3 \\ 1.2 & 0.8 \end{pmatrix}$$

(a) *Is the origin an attractor, repeller, or saddle point?*

(b) *Find the directions of greatest attraction or repulsion.*

SOLUTION

Solution to (a). The origin is a $\boxed{\text{saddle point.}}$

To see this, we compute that the characteristic polynomial is

$$p_A(t) = \det \begin{pmatrix} t-1.7 & -0.3 \\ -1.2 & t-0.8 \end{pmatrix} = (t^2 - 2.5t + 1.36) - (.36) = t^2 - 2.5t + 1$$

$$= (t-2)(t-\frac{1}{2})$$

Thus the eigenvalues are $\lambda = \frac{1}{2}, 2$. Since $0 < \frac{1}{2} < 1$ and $1 < 2$, we see that the origin is a saddle point. □

Solution to (b). We have that the line spanned by $\boxed{\begin{pmatrix} 1 \\ -4 \end{pmatrix}}$ is the direction of greatest attraction, and the line spanned by $\boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ is the direction of greatest repulsion.

To deduce this, we find the eigenspaces. We start with the $\lambda = \frac{1}{2}$-eigenspace, $E_{1/2}$, which is the kernel of $\frac{1}{2}I - A$:

$$\frac{1}{2}I - A = \begin{pmatrix} -1.2 & -0.3 \\ -1.2 & -0.3 \end{pmatrix} \mapsto \begin{pmatrix} 12 & 3 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1/4 \\ 0 & -1 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is a basis for the $\frac{1}{2}$-eigenspace $E_{1/2}$.

We now compute the $\lambda = 2$-eigenspace, E_2, which is the kernel of $2I - A$:

$$2I - A = \begin{pmatrix} 0.3 & -0.3 \\ -1.2 & 1.2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ \color{red}{0} & \color{blue}{-1} \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for the 2-eigenspace E_2.

In conclusion, the line spanned by $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is the direction of greatest attraction, and the line spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the direction of greatest repulsion. □

8. (10 points) • **TRUE** or **FALSE**:

(a) Suppose A and B are invertible $n \times n$ matrices, and that $AB = BA$. Then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

TRUE: $(AB)^{-1} = B^{-1}A^{-1}$ and $(BA)^{-1} = A^{-1}B^{-1}$.

(b) Let $f : V \to V$ be a linear map of a vector space to itself. If f is surjective, then f is an isomorphism.

FALSE: We have seen examples where this fails. If V were assumed to be finite dimensional, however, then this statement would be true.

(c) Suppose that P is an $n \times n$ matrix with positive entries, such that the column sums are equal to 1. Then $\lim_{n\to\infty} P^n$ exists.

TRUE: We have seen this in class. (This is also Lay, Section 4.9, p.261, Theorem 18.)

(d) Suppose that $T : V \to V'$ is a linear map of finite dimensional vector spaces. Then $\dim V' = \dim \ker(T) + \dim \operatorname{Im}(T)$.

FALSE: Take $V = \mathbb{R}$ and $V' = 0$. (The Rank–Nullity Theorem states that $\dim V = \dim \ker(T) + \dim \operatorname{Im}(T)$.)

(e) If an $n \times n$ matrix has n distinct eigenvalues, then it has n linearly independent eigenvectors.

TRUE: We have seen this in class. (This is also Lay, Section 5.3, p.286, Theorem 6, combined with p.284 Theorem 5.)

(f) If v is an eigenvector for an $n \times n$ matrix A with eigenvalue λ, and $r \neq 0$ is a real number, then rv is an eigenvector for A with eigenvalue λ.

TRUE: $A(rv) = rAv = r\lambda v = \lambda(rv)$.

(g) Suppose that M is an $n \times n$ matrix and $M^N = 0$ for some integer $N > 1$. Then M is diagonalizable.

FALSE: The matrix $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $M^2 = 0$, but M is not diagonalizable. (Note more generally that if $M = S^{-1}DS$ for a diagonal matrix D, then $0 = M^n = S^{-1}D^nS$ if and only if $D = 0$ (and hence $M = 0$), since S and S^{-1} induce isomorphisms.)

(h) For an $n \times n$ matrix A, if $\det(\operatorname{cof} A) = 0$, then $\det A = 0$.

TRUE: We know that $A(\operatorname{cof} A)^T = (\det A)I$, so that $0 = (\det A)(\det(\operatorname{cof} A)) = (\det A)(\det((\operatorname{cof} A)^T)) = (\det A)^n$.

(i) If V is a real vector space, and $W, W' \subseteq V$ are real vector subspaces of V, then $W \cap W'$ is a real vector subspace of V.

TRUE: We have seen this is class.

(j) The row space of a matrix is the same as the row space of the reduced row echelon form of the matrix.

TRUE: We saw this in class.