Midterm 1

Linear Algebra

MATH 2130

Spring 2021

Friday February 12, 2021

NAME: **Enter your name here** ______________________________

PRACTICE EXAM

SOLUTIONS

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

- This exam is closed book.
- You may use only paper and pencil.
- You may not use any other resources whatsoever.
- You will be graded on the clarity of your exposition.

1. (10 points) • **TRUE** or **FALSE**: *Suppose that $V \subseteq \mathbb{R}^n$ is a nonempty subset satisfying:*

1. *For all $v_1, v_2 \in V$, we have $v_1 + v_2 \in V$.*
2. *For all $v \in V$, we have $-v \in V$.*

Then V is a subspace of $\mathbb{R}^n$.

If true, state this clearly at the start of your solution, and provide a proof. If false, state this clearly at the start of your solution, provide a counterexample, and prove that it is a counterexample.

SOLUTION

Solution. **FALSE** For instance, the set $\mathbb{Z}^n \subseteq \mathbb{R}^n$, i.e., the set of elements of $\mathbb{R}^n$ with integral coordinates, is a counterexample to the statement. In other words, I claim that $\mathbb{Z}^n$ is a nonempty subset of $\mathbb{R}^n$ that satisfies both 1. and 2. above, but is not a subspace of $\mathbb{R}^n$.

To see that the subset $\mathbb{Z}^n$ is nonempty, we can just observe that $(0, \dots, 0)$ is an element of $\mathbb{Z}^n$.

To see that the subset $\mathbb{Z}^n$ of $\mathbb{R}^n$ satisfies 1. above, we can argue as follows. Suppose that $(z_1, \dots, z_n)$ and $(w_1, \dots, w_n)$ are elements of $\mathbb{Z}^n$. Then

$$(z_1, \dots, z_n) + (w_1, \dots, w_n) = (z_1 + w_1, \dots, z_n + w_n) \in \mathbb{Z}^n,$$

since the sum of any two integers is an integer.

To see that the subset $\mathbb{Z}^n$ of $\mathbb{R}^n$ satisfies 2. above, we can argue as follows. Suppose that $(z_1, \dots, z_n)$ is an element of $\mathbb{Z}^n$. Then

$$-(z_1, \dots, z_n) = (-z_1, \dots, -z_n) \in \mathbb{Z}^n,$$

since the negative of any integer is an integer.

Finally, we have that $\mathbb{Z}^n$ is not a subspace of $\mathbb{R}^n$, since, for example, $(1, \dots, 1) \in \mathbb{Z}^n$, but $\frac{1}{2}(1, \dots, 1) \notin \mathbb{Z}^n$. $\square$

2. (10 points) • *Find all solutions to the following system of linear equations:*

$$\begin{array}{rcrcrcr} 3x_1 & + & 9x_2 & + & 27x_3 & = & -3 \\ -3x_1 & - & 11x_2 & - & 35x_3 & = & 5 \\ 2x_1 & + & 8x_2 & + & 26x_3 & = & -4 \end{array}$$

SOLUTION

Solution. The solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \left\{ \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

To find this, we row reduce the associated augmented matrix

$$\left[\begin{array}{rrr|r} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{array}\right]$$

$$\begin{array}{r} R_1' = \frac{1}{3}R_1 \\ \\ R_3' = \frac{1}{2}R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 3 & 9 & -1 \\ -3 & -11 & -35 & 5 \\ 1 & 4 & 13 & -2 \end{array}\right]$$

$$\begin{array}{r} \\ R_2' = 3R_1 + R_2 \\ R_3' = -R_1 + R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 3 & 9 & -1 \\ 0 & -2 & -8 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right]$$

$$\begin{array}{lcr} & & \\ R_2' = R_3 & \mapsto & \\ R_3' = R_2' & & R_3'' = 2R_2 + R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 3 & 9 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} R_1' = R_1 - 3R_2 \\ \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now we adjust the RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Thus the solutions to the system of equations are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \left\{ \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

as claimed.

To check your answer, you can confirm that these are in fact solutions; e.g., $(x_1, x_2, x_3) = (2, -1, 0)$ is a solution to the system of equations, and $(x_1, x_2, x_3) = (-3, 4, -1)$ is a solution to the following homogeneous system of equations:

$$\begin{array}{rcrcrcl} 3x_1 & + & 9x_2 & + & 27x_3 & = & 0 \\ -3x_1 & - & 11x_2 & - & 35x_3 & = & 0 \\ 2x_1 & + & 8x_2 & + & 26x_3 & = & 0 \end{array}$$

□

3. (10 points) • Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{bmatrix}$$

(a) *Find the reduced row echelon form of A.*

(b) *Are the columns of A linearly independent?*

(c) *Are the rows of A linearly independent?*

(d) *What is the column rank of A?*

(e) *What is the row rank of A?*

SOLUTION

Solution. (a) The RREF of the matrix A is

$$\boxed{\mathrm{RREF}(A) = \begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Indeed we have

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{bmatrix}$$

$$\begin{matrix} \\ \\ R_3' = -3R_1 + R_3 \\ R_4' = -R_1 + R_4 \end{matrix} \begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \\ \\ R_3' = -\frac{1}{10}R_3 \\ R_4' = -R_2 + R_4 \end{matrix} \begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} R_1' = R_1 - 4R_3 \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The columns of A are **not linearly independent** since at most 4 vectors in $\mathbb{R}^4$ can be linearly independent.

(c) The rows of A are **not linearly independent**, since RREF(A) has a zero row.

(d) The column rank is equal to the row rank, which is $\boxed{3}$, the number of nonzero rows in the RREF of A.

(e) The row rank is $\boxed{3}$. □

4. (10 points) • Consider the linear map $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L(x_1, x_2, x_3) = (2x_1 - x_3, 3x_2 + x_3).$$

Write down the matrix form of the linear map L.

SOLUTION

Solution. The matrix form of L is

$$\boxed{\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}}$$

We find this by computing L on the standard basis elements:

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 1 - 0 \\ 3 \cdot 0 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 0 - 0 \\ 3 \cdot (1) + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 0 - 1 \\ 3 \cdot 0 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These give the corresponding columns of the matrix form of L. □

5. (10 points) • Consider the matrix

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) *Find the inverse of B.*

(b) *Does there exist* $x \in \mathbb{R}^3$ *such that* $Bx = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$?

SOLUTION

Solution. (a) The solution is

$$B^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -3 & -1 & 6 \end{bmatrix}$$

To do this, we consider the augmented matrix $\left[\begin{array}{c|c} B & I \end{array}\right]$, and do row reduction until we arrive at the matrix $\left[\begin{array}{c|c} I & B^{-1} \end{array}\right]$. In more detail:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -6 & -1 & -3 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -6 & -1 & -3 & 1 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & -6 \end{array}\right]$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & -1 & 6 \end{array}\right]$$

The matrix on the right is the matrix B^{-1}.

You can check your answer by computing:

$$BB^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -3 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) **YES** Since B is invertible, given any $b \in \mathbb{R}^3$, we have that $B(B^{-1}b) = b$. In particular, for $b = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$, we have that $x = B^{-1}\begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$ satisfies $Bx = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$.

□

6. (10 points) • **TRUE** or **FALSE**:

(a) Let $A \in M_{m\times n}(\mathbb{R})$. There is an $x \in \mathbb{R}^n$ such that $Ax = 0$.

TRUE: Take $x = 0$.

(b) Let $A \in M_{m\times n}(\mathbb{R})$. If the columns of A span $\mathbb{R}^m$, then for any $b \in \mathbb{R}^m$ there is an $x \in \mathbb{R}^n$ such that $Ax = b$.

TRUE: The image of the linear map is the column span.

(c) The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$ is a linear map.

FALSE: $f(1) + f(1) \neq f(2)$.

(d) If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{bmatrix}$ for each natural number n.

FALSE: $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

(e) If A and B are $m \times n$ matrices, then $A + B = B + A$.

TRUE: We have $(A + B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B + A)_{ij}$.

(f) Let $A \in M_{m\times n}(\mathbb{R})$. If the rows of A are linearly independent, then for any $b \in \mathbb{R}^m$ there is at most one $x \in \mathbb{R}^n$ such that $Ax = b$.

FALSE: Take $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $b = 0$.

(g) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The kernel of f is a sub-vector space of $\mathbb{R}^n$.

TRUE: We have seen this in class.

(h) If the columns of a square matrix A are linearly independent, then A^T is invertible.

TRUE: This follows from our characterization of invertible matrices: A^T invertible $\iff$ A invertible $\iff$ columns of A are linearly independent.

(i) If $V, W \subseteq \mathbb{R}^n$ are subspaces. The union $V \cup W$ is a subspace of $\mathbb{R}^n$.

FALSE: Take $V = \text{Span}((1, 0))$ and $W = \text{Span}((0, 1))$ in $\mathbb{R}^2$.

(j) Suppose that A and B are square matrices, and AB is invertible. Then A and B are invertible.

TRUE: If AB is invertible, then as a linear maps, B is injective and A is surjective, which we have seen, for square matrices, is enough to show that the matrices are invertible.