FINAL EXAM
LINEAR ALGEBRA

MATH 2135
SUMMER 2018

Friday July 6, 2018
9:15 AM – 10:50 AM

Name

PRACTICE EXAM

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	6	7	8	9	
10	10	10	10	10	10	10	10	10	10 total

Date: July 2, 2018.

1. Find the determinant of each of the following matrices.

1
10 points

1.(a). $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

1.(b). $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}$

2
10 points

2. Let $V = \mathbb{R}[x]$ be the vector space of real polynomial functions. Let

$$D : V \to V$$

be the derivative map; i.e. $D(p) = p'$ for all $p \in V$. Let

$$E : V \to V$$

be the integration map that sends a polynomial p to the polynomial q given by $q(x) = \int_0^x p(t)dt$, for all $x \in \mathbb{R}$. It is a fact that D and E are linear maps.

2.(a). *Show that D is surjective, but not injective.*

2.(b). *Show that E is injective, but not surjective.*

3. Suppose we have a two state Markov chain with stochastic matrix

3
10 points

$$P = \begin{pmatrix} 0.1 & 0.5 \\ 0.9 & 0.5 \end{pmatrix}$$

Given the probability vector $v = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$, *find* $\lim_{n\to\infty} P^n v$.

4. Consider the following matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

4
10 points

4.(a). *Find the characteristic polynomial* $p_A(t)$ *of* A.

4.(b). *Find the eigenvalues of* A.

4.(c). *Find an orthonormal basis for each eigenspace of* A *in* $\mathbb{R}^3$.

4.(d). *Is* A *diagonalizable? If so, find a matrix* $S \in \mathrm{M}_{3\times3}(\mathbb{R})$ *so that* $S^{-1}AS$ *is diagonal. If not, explain.*

4.(e). *Is* A *diagonalizable with orthogonal matrices? If so, find an orthogonal matrix* $U \in \mathrm{M}_{3\times3}(\mathbb{R})$ *so that* U^TAU *is diagonal. If not, explain.*

5. Consider the following matrix:

5
10 points

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}$$

5.(a). *What is the sum of the roots of the characteristic polynomial of B?*

5.(b). *What is the product of the roots of the characteristic polynomial of B?*

5.(c). *Are the roots of the characteristic polynomial of B real?*

6. Consider the two dimensional discrete dynamical system

6
10 points

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{pmatrix} 1.7 & 0.3 \\ 1.2 & 0.8 \end{pmatrix}$$

6.(a). *Is the origin an attractor, repeller, or saddle point?*

6.(b). *Find the directions of greatest attraction or repulsion.*

7
10 points

7. Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Find an orthonormal basis for the vector subspace of $\mathbb{R}^4$ *spanned by* $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, *and* $\mathbf{x}_4$.

8
10 points

8. *Find the equation* $y = \beta_0 + \beta_1 x$ *of the line that best fits the given data points, as a least squares model:*

$$\begin{bmatrix} x \\ y \end{bmatrix}: \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

9. **TRUE** or **FALSE**. You do **not** need to justify your answer.

9
10 points

9.(a). Suppose A and B are invertible $n \times n$ matrices, and that $AB = BA$. Then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

T F

9.(b). Let $f : V \to V$ be a linear map of a vector space to itself. If f is surjective, then f is an isomorphism.

T F

9.(c). Suppose that P is an $n \times n$ matrix with positive entries, such that the column sums are equal to 1. Then $\lim_{n\to\infty} P^n$ exists.

T F

9.(d). Suppose that $T : V \to V'$ is a linear map of finite dimensional vector spaces. Then $\dim V' = \dim \ker(T) + \dim \operatorname{Im}(T)$.

T F

9.(e). If an $n \times n$ matrix has n distinct eigenvalues, then it has n linearly independent eigenvectors.

T F

9.(f). If v is an eigenvector for an $n \times n$ matrix A with eigenvalue λ, and $r \neq 0$ is a real number, then rv is an eigenvector for A with eigenvalue λ.

T F

9.(g). Suppose that $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ is symmetric, and let $v_1, v_2 \in \mathbb{R}^n$ be eigenvectors with corresponding eigenvalues λ_1, λ_2. If $\lambda_1 \neq \lambda_2$, then v_1 is orthogonal to v_2.

T F

9.(h). Suppose that M is an $n \times n$ matrix and $M^N = 0$ for some integer $N > 1$. Then M is diagonalizable.

T F

9.(i). For an $n \times n$ matrix A, if $\det(\operatorname{cof} A) = 0$, then $\det A = 0$.

T F

9.(j). Let $v, w \in \mathbb{R}^n$. If θ is the angle between v and w, then $\cos\theta = \dfrac{v.w}{||v||||w||}$.

T F