FINAL EXAM
LINEAR ALGEBRA

MATH 2135

Monday May 7, 2018
1:30 PM – 3:30 PM

Name

PRACTICE EXAM

SOLUTIONS

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	6	7	8	9	10	
10	10	10	10	10	10	10	10	10	10	10 total

Date: May 6, 2018.

1. *Give the definition of a vector space.*

1
10 points

SOLUTION

See the pdf online:
`www.math.colorado.edu/~casa/teaching/18spring/2135/hw/LinAlgHW.pdf`

2. Consider the following matrix

2
10 points

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

2.(a). *Find the characteristic polynomial $p_A(t)$ of A.*

2.(b). *Find the eigenvalues of A.*

2.(c). *Find an orthonormal basis for each eigenspace of A in $\mathbb{C}^3$.*

2.(d). *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{C})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

2.(e). *Is A diagonalizable with unitary matrices? If so, find a unitary matrix $U \in \mathrm{M}_{3\times 3}(\mathbb{C})$ so that U^*AU is diagonal. If not, explain.*

SOLUTION

(a) We have

$$\begin{aligned} p_A(t) &= \begin{vmatrix} t-2 & 1 & -1 \\ 0 & t-3 & 1 \\ -2 & -1 & t-3 \end{vmatrix} \\ &= (t-2)[(t-3)^2 - (1)(-1)] - (1)[0-(1)(-2)] + (-1)[0-(t-3)(-2)] \\ &= (t-2)[t^2-6t+10] - 2 + \underbrace{(t-3)(-2)}_{-2t+6} \\ &= (t^3 - 6t^t + 10t - 2t^2 + 12t - 20) - 2 + (6-2t) \\ &= t^3 - 8t^2 + 20t - 16. \end{aligned}$$

In other words, the solution is:

$$\boxed{p_A(t) = t^3 - 8t^2 + 20t - 16.}$$

As a quick partial check of the solution, observe that

$$\mathrm{tr}(A) = 8$$

$$\det A = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 2 & 2 \end{vmatrix} = 2(6+2) = 16.$$

confirming the computation of the coefficients of t^2 and t^0.

(b) One can easily check that

$$p_A(2) = 2^3 - 8 \cdot 2^2 + 20 \cdot 2 - 16 = 8 - 32 + 40 - 16 = 48 - 48 = 0.$$

Thus we have

$$p_A(t) = (t-2)(t^2 - 6t + 8) = (t-2)(t-2)(t-4).$$

Thus the eigenvalues are

$$\boxed{\lambda = 2, 4.}$$

(c) To find the $\lambda = 2$ eigenspace E_2, we compute

$$E_2 := \ker(2I - A) = \ker \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -1 & -1 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We add rows, and get the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus we have

$$E_2 = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

Now we compute the $\lambda = 4$ eigenspace E_4. We have

$$E_4 = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus we have

$$E_4 = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

Thus the solution to the problem is:

The eigenspaces for A are E_2 and E_4, and we have that $\boxed{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$ *is a basis for E_2 and* $\boxed{\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}$ *is a basis for E_4.*

Note that we can easily double check that the given basis elements are eigenvectors.

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+1-1 \\ -3+1 \\ 2-1-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-1-1 \\ 3+1 \\ -2+1-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$$

(d) $\boxed{\text{No.}}$ A is not diagonalizable since $\mathbb{C}^3$ does not admit a basis of eigenvectors for A.

(e) $\boxed{\text{No.}}$ A is not diagonalizable with unitary matrices either, since it is not even diagonalizable.

3. Consider the following matrix

3
10 points

$$B = \begin{pmatrix} 1 & 2 & 0 & 2 & -1 & 0 \\ 3 & -1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix}$$

3.(a). *What is the sum of the roots of the characteristic polynomial of B?*

3.(b). *What is the product of the roots of the characteristic polynomial of B?*

3.(c). *Does B admit an orthonormal basis of eigenvectors in $\mathbb{R}^6$?*

SOLUTION

(a) The sum of the roots of the characteristic polynomial of B is equal to the trace of B. So we have

$$\operatorname{tr} B = 1 + (-1) + 2 + 1 + 8 + 0 = 11.$$

So the answer is $\boxed{11}$.

(b) The product of the roots of the characteristic polynomial of B is equal to the determinant of B (since it is a 6×6 matrix). Since B is block-upper-triangular, we could compute the determinant that way; but the fourth and fifth rows are linearly dependent, so the determinant is 0. Thus the answer is $\boxed{0}$.

(c) $\boxed{\text{No.}}$ A real matrix in $\mathbb{R}^n$ admits an orthonormal basis of eigenvectors if and only if it is symmetric. That a symmetric matrix admits an orthonormal basis of eigenvectors is the Spectral Theorem. The converse is elementary: if $A = U^T D U$ for U a real orthogonal matrix and D a real diagonal matrix, then $A^T = (U^T D U)^T = U^T D^T U = U^T D U = A$.

4
10 points

4. Suppose that $(V_1, +_1, \cdot_1)$ and $(V_2, +_2, \cdot_2)$ are K-vector spaces. Define maps:

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \to V_1 \times V_2$$
$$(v_1, v_2) + (v_1', v_2') = (v_1 +_1 v_1', v_2 +_2 v_2')$$

and

$$\cdot : K \times (V_1 \times V_2) \to V_1 \times V_2$$
$$\lambda \cdot (v_1, v_2) = (\lambda \cdot_1 v_1, \lambda \cdot_2 v_2).$$

Show that the triple $(V_1 \times V_2, +, \cdot)$ is a K-vector space.

We denote this vector space by $V_1 \times V_2$ or $V_1 \oplus V_2$, and call it the direct product, or direct sum, respectively, of the vector spaces V_1 and V_2.

SOLUTION

We check that the triple satisfies the definition of a vector space. Below is a sketch.

Condition (1)(a): the additive identity is $(\mathscr{O}_1, \mathscr{O}_2)$ where $\mathscr{O}_i$ is the identity element of V_i, $i = 1, 2$. Indeed, if $(v_1, v_2) \in V_1 \times V_2$, then $(v_1, v_2) + (\mathscr{O}_1, \mathscr{O}_2) = (v_1 + \mathscr{O}_1, v_2 + \mathscr{O}_2) = (v_1, v_2)$.

Condition (1)(b): if $(v_1, v_2) \in V_1 \times V_2$, then its additive inverse is $(-v_1, -v_2)$. I leave the details for you to check.

Condition (1)(c): given $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V_1 \times V_2$, then

$$\begin{aligned}
((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\
&= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2) \\
&= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)) \\
&= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \\
&= (x_1, x_2) + ((y_1, y_2) + (z_1, z_2)).
\end{aligned}$$

Condition (2)(a): Given $(x_1, x_2), (y_1, y_2)$, one shows that

$$(x_1, x_2) + (y_1, y_2) = (y_1, y_2) + (x_1, x_2).$$

This follows from the abelian property for V_1 and V_2. I leave the details for you to check.

Conditions (3)(a)–(d) are similar, and I leave the details for you to check.

5
10 points

5. Suppose that W is a finite dimensional subspace of a Euclidean space $(V, (-,-))$. Suppose that $L : V \to V$ is a linear map such that

- $\operatorname{Im} L = W$.
- $L \circ L = L$.
- $\ker L = W^\perp$.

Show that if $e_1, \ldots, e_n$ form an orthonormal basis for W, then L is given by

$$L(v) = \sum_{i=1}^{n} (v, e_i) e_i.$$

In other words, show that L is the orthogonal projection onto W.

SOLUTION

We will show that $L(v) = \sum_{i=1}^{n} (v, e_i) e_i$ in the following way:

(1) First we will show that *there is a unique linear map $L : V \to V$ satisfying the three bullet points*. In other words, if $L, L' : V \to V$ are two linear maps that both satisfy the three bullet points, then $L = L'$.
(2) Then we will show that the map $L' : V \to V$ given by $L'(v) = \sum_{i=1}^{n} (v, e_i) e_i$ *satisfies the three bullet points*.

(1) Let V be a linear map satisfying the three bullet points.

The first two bullet points imply that L acts by the identity on W. Indeed, if $w \in W$, then $w = L(v)$ for some $v \in V$, by the first bullet point. Then we have $L(w) = L(L(v)) = L(v) = w$, by the second bullet point. Thus L acts by the identity on W.

On the other hand, we have shown in a theorem that every element v in V can be written uniquely as $v = w + w^\perp$ with $w \in W$ and $w^\perp \in W^\perp$. Thus $L(v) = L(w + w^\perp) = L(w) + L(w^\perp) = w$ by the third bullet point. Since $L(v) = w$ for any linear map L satisfying the three bullet points, such a linear map is unique.

Note also that $L(v) = w$ is how we defined the orthogonal projection in class, and so we have just proved that L is the orthogonal projection.

We also proved in class that the orthogonal projection was given in terms of an orthonormal basis by the formula above. But for completeness, let us establish (2).

(2) Let L' be the linear map defined by $L'(v) = \sum (v, e_i) e_i$. We clearly have that $\operatorname{Im}(L') \subseteq W$. Now let $w \in W$. We can write $w = \sum_{j=1}^{n} \alpha_j e_j$, for some $\alpha_j \in \mathbb{C}$, since $e_1, \ldots, e_n$ form a basis for W. Taking $v = w$, one has $L'(v) = \sum_{i=1}^{n} (v, e_i) e_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} \alpha_j e_j, e_i) e_i = \sum_{i=1}^{n} \alpha_i (e_i, e_i) e_i = w$. This shows $\operatorname{Im}(L') \supseteq W$, so that $\operatorname{Im}(L') = W$, establishing the first bullet point.

For the second bullet point, we have

$$L'(L'(v)) = L'(\sum_{i=1}^{n}(v,e_i)e_i) = \sum_{i=1}^{n}(v,e_i)L'(e_i) = \sum_{i=1}^{n}(v,e_i)\sum_{j=1}^{n}(e_i,e_j)e_j = \sum_{i=1}^{n}(v,e_i)e_i = L'(v).$$

Finally, let us show the third bullet point, that $\ker L' = W^{\perp}$. We start with the observation

$$v \in \ker L' \iff 0 = L'(v) = \sum_{i=1}^{n}(v,e_i)e_i \iff (v,e_i) = 0, \quad i = 1,\ldots,n.$$

The last $\iff$ above comes from the fact that the e_i are linearly independent. The claim is that

$$(v,e_i) = 0, \quad i = 1,\ldots,n \iff v \in W^{\perp}.$$

The implication $\Longleftarrow$ follows from the definition of $W^{\perp}$. For the opposite implication, suppose $(v,e_i) = 0, \quad i = 1,\ldots,n$ and $w \in W$. Then $w = \sum_{i=1}^{n}\alpha_i e_i$, and we have $(v,w) = (v,\sum_{i=1}^{n}\alpha_i e_i) = \sum_{i=1}^{n}\bar{\alpha}_i(v,e_i) = 0$.

Remark 0.1. If instead L satisfies only two out of the three bullet point conditions, must L be the orthogonal projection onto W? The answer is no, and I'll leave it to you to find examples for each of the three possible choices of two conditions.

6
10 points

6. Let $(V, (-,-))$ be a Euclidean space and suppose that $L : V \to V$ is a linear map admitting an adjoint L^*. *If $L = L^*$, show that all the eigenvalues of L are real.*

SOLUTION

For a real Euclidean space there is nothing to show. For a complex Euclidean space it suffices to show that every eigenvalue is equal to its complex conjugate. So let v be an eigenvector with eigenvalue λ.

$$\lambda = \frac{(\lambda v, v)}{(v,v)} = \frac{(L(v), v)}{(v,v)} = \frac{(v, L^*(v))}{(v,v)} = \frac{(v, L(v))}{(v,v)} = \frac{(v, \lambda v)}{(v,v)} = \bar{\lambda}.$$

7. *Show that an $m \times n$ matrix, with $m \leq n$, has rank m if and only if it has an $m \times m$ minor with non-zero determinant.*

7
10 points

SOLUTION

Suppose first that our matrix $A \in M_{m\times n}(K)$ has rank $< m$. Then there is a linear relation among the rows of A. Any $m \times m$ minor of A is obtained by removing $n - m$ columns of A. There is clearly still a relation among the rows of the minor, and so the determinant of the minor is 0.

Conversely, suppose that our matrix $A \in M_{m\times n}(K)$ has rank m. Then there exist m columns of A that are linearly independent. The $m \times m$ minor of A obtained using those m columns then has rank m, and so has non-zero determinant.

8
10 points

8. In this problem we will work with matrices $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$.

8.(a). We say that A is similar to B, and write $A \sim B$, if there is an invertible matrix $S \in \mathrm{M}_{n\times n}(\mathbb{C})$ such that $B = S^{-1}AS$. *Show that $\sim$ defines an equivalence relation on* $\mathrm{M}_{n\times n}(\mathbb{C})$.

8.(b). *Show that any $A \in \mathrm{M}_{n\times n}(\mathbb{C})$ is similar to an upper triangular matrix.*

8.(c). Suppose that $A \in \mathrm{M}_{2\times 2}(\mathbb{C})$ has distinct eigenvalues λ_1 and λ_2. *Show that A is similar to a diagonal matrix with λ_1 and λ_2 on the diagonal.*

8.(d). Suppose that $A \in \mathrm{M}_{2\times 2}(\mathbb{C})$ has a single eigenvlaue λ. *Show that A is similar to either λI or to a matrix of the form*

$$J_\lambda := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

and that λI is not similar to J_λ.

8.(e). *Use the previous parts of the problem to describe the equivalence classes of matrices in* $\mathrm{M}_{2\times 2}(\mathbb{C})$ *under the equivalence relation $\sim$.*

SOLUTION

(a) First let us show that for any $A \in \mathrm{M}_{n\times n}(\mathbb{C})$, we have $A \sim A$. For this, take $S = I$. Next let us show that if $A \sim B$, then $B \sim A$. Well, if $A \sim B$, then there is an invertible matrix S such that $B = S^{-1}AS$. Then $A = SBS^{-1} = (S')^{-1}BS'$ with $S' = S^{-1}$, so $B \sim A$. Now suppose that $A \sim B$ and $B \sim C$. Then we have invertible matrices S and T with $B = S^{-1}AS$ and $C = T^{-1}BT$. Together we have

$$C = T^{-1}BT = T^{-1}S^{-1}AST = (S')^{-1}AS'$$

with $S' = ST$. Thus $A \sim C$.

(b) We will prove this by induction on n. The case with $n = 1$ is obvious. So suppose we have $A \in \mathrm{M}_{n\times n}(\mathbb{C})$. For each of the up to n distinct eigenvalues, there is an eigenvector. In particular, there is at least one eigenvalue, say λ_1, and so there exists an eigenvector v_1 with eigenvalue λ_1. Extend this to a basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ arbitrarily. Let S be the matrix obtained by taking these vectors as columns. Then we have

$$B := S^{-1}AS = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

Now consider the $(n-1) \times (n-1)$ matrix

$$B' = \begin{pmatrix} b_{22} & \cdots & b_{2n} \\ \vdots & \ddots & \vdots \\ b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

By induction, there exists a basis $v_2', \ldots, v_n'$ of $\mathbb{C}^{n-1}$ such that the matrix S' given by taking those vectors as columns satisfies

$$C' := S'^{-1} B' S' = \begin{pmatrix} c_{22} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{nn} \end{pmatrix}$$

is upper triangular. View the vectors $v_2', \ldots, v_n'$ as vectors in $\mathbb{C}^n$ by taking the first coordinate to be 0. The vectors $e_1, v_2', \ldots, v_n'$ then form a basis of $\mathbb{C}^n$. Let T be the matrix obtained by taking these basis vectors as the columns. We have

$$C := T^{-1} B T = \begin{pmatrix} \lambda_1 & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix}$$

is upper triangular. Thus we have $C = T^{-1} S^{-1} A S T$ is upper triangular.

(c) This is a special case of a theorem we proved about $n \times n$ matrices.

(d) Suppose that $A \in \mathrm{M}_{2\times 2}(\mathbb{C})$ has a single eigenvlaue λ. We know that A is similar to an upper triangular matrix

$$T = \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}$$

If $t = 0$, we have the matrix λI. If $t \neq 0$, we can compute

$$\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & -b/d \\ 0 & 1/d \end{pmatrix} = \begin{pmatrix} \lambda & t/d \\ 0 & \lambda \end{pmatrix}$$

Thus taking $d = t$, we have that T, and therefore A, is similar to a matrix of the form

$$J_\lambda := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We can see that λI is not similar to J_λ, by considering the dimension of E_λ, the λ-eigenspace; in the first instance $\dim E_\lambda = 2$, while in the second $\dim E_\lambda = 1$.

(e) The previous parts of the problem imply that every matrix $A \in M_{2\times 2}(\mathbb{C})$ is similar to a matrix of the form:

- $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{C}$, or,
- $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{C}$.

Moreover distinct matrices of the form above are similar if and only if they are both of the type in the first bullet point, and have the same entries on the diagonal, but in opposite order.

More precisely, let $\mathbb{C}^2/\Sigma_2$ be the set of equivalence classes of elements in $\mathbb{C}^2$, under the equivalence relation $(\lambda_1, \lambda_2) \sim (\lambda_2, \lambda_1)$. Then we have a bijection

$$\mathbb{C}^2/\Sigma_2 \sqcup \mathbb{C} \xleftrightarrow{1:1} M_{2\times 2}(\mathbb{C})/\sim$$
$$[(\lambda_1, \lambda_2)] \mapsto \left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right]$$
$$\lambda \mapsto \left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right].$$

In other words, the set of equivalence classes of matrices in $M_{2\times 2}(\mathbb{C})$ is in bijection with the disjoint union of $\mathbb{C}^2/\Sigma_2$ with $\mathbb{C}$.

9. In this problem we will work with matrices $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$.

9
10 points

9.(a). We say that A is unitarily similar to B, and write $A \sim_U B$ if there is a unitary matrix $U \in \mathrm{M}_{n\times n}(\mathbb{C})$ such that $B = U^*AU$. *Show that* $\sim_U$ *defines an equivalence relation on* $\mathrm{M}_{n\times n}(\mathbb{C})$.

9.(b). *Show that any* $A \in \mathrm{M}_{n\times n}(\mathbb{C})$ *is unitarily similar to an upper triangular matrix.*

9.(c). Suppose that $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$ are *upper triangular*, with the same diagonal entries $a_{ii} = b_{ii}$, $i = 1, \ldots, n$, with $a_{ii} \neq a_{jj}$, $i \neq j$. *If* $U \in M_{n\times n}(\mathbb{C})$ *is a unitary matrix such that* $B = U^*AU$, *then show that* U *is diagonal.*

9.(d). Suppose $T \in \mathrm{M}_{2\times 2}(\mathbb{C})$ is an upper triangular matrix:

$$T = \begin{pmatrix} \lambda_1 & t_{12} \\ 0 & \lambda_2 \end{pmatrix}.$$

Show that

$$|t_{12}|^2 = \mathrm{tr}(T^*T) - |\lambda_1|^2 - |\lambda_2|^2.$$

9.(e). Suppose that $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$ are unitarily similar. *Show that* $\mathrm{tr}(A^*A) = \mathrm{tr}(B^*B)$.

SOLUTION

(a) First let us show that for any $A \in \mathrm{M}_{n\times n}(\mathbb{C})$, we have $A \sim_U A$. For this, take $U = I$. Next let us show that if $A \sim_U B$, then $B \sim_U A$. Well, if $A \sim_U B$, then there is a unitary matrix U such that $B = U^*AU$. Then $A = UBU^* = (U')^*BU'$ with $U' = U^*$, so $B \sim_U A$. Now suppose that $A \sim_U B$ and $B \sim_U C$. Then we have unitary matrices U and V with $B = U^*AU$ and $C = V^*BV$. Together we have

$$C = V^*BV = V^*U^*AUV = (U')^*AU'$$

with $U' = UV$. Thus $A \sim_U C$.

(b) We will prove this by induction on n. The case with $n = 1$ is obvious. So suppose we have $A \in \mathrm{M}_{n\times n}(\mathbb{C})$. For each of the up to n distinct eigenvalues, there is an eigenvector. In particular, there exists an eigenvector v_1, with eigenvalue say λ_1. Take $u_1 = v_1/||v_1||$, and extend this to an orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{C}^n$ arbitrarily. Let U be the matrix obtained by taking these vectors as columns. Then we have

$$B := U^*AU = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

Now consider the $(n-1)\times(n-1)$ matrix

$$B' = \begin{pmatrix} b_{22} & \cdots & b_{2n} \\ \vdots & \ddots & \vdots \\ b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

By induction, there exists an orthonormal basis $u'_2, \ldots, u'_n$ of $\mathbb{C}^{n-1}$ such that the matrix U' given by taking those vectors as columns satisfies

$$C' := U'^* B' U' = \begin{pmatrix} c_{22} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{nn} \end{pmatrix}$$

is upper triangular. View the vectors $u'_2, \ldots, u'_n$ as orthonormal vectors in $\mathbb{C}^n$ by taking the first coordinate to be 0. The vectors $e_1, u'_2, \ldots, u'_n$ then form an orthonormal basis of $\mathbb{C}^n$. Let V be the matrix obtained by taking these basis vectors as the columns. We have

$$C := V^* B V = \begin{pmatrix} \lambda_1 & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix}$$

is upper triangular. Thus we have $C = V^* U^* A U V$ is upper triangular.

(c) Here I will just give a sketch of the proof. We have

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ & \ddots & \vdots \\ & & b_{nn} \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$$

with $B = U^* A U$, or in other words $UB = AU$, or in matrix form:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ & \ddots & \vdots \\ & & b_{nn} \end{pmatrix}$$

Considering the $n,1$ entry (bottom left corner), we have $a_{nn} u_{n1} = u_{n1} b_{11}$. Then since $a_{nn} \neq b_{11}$, we have $u_{n1} = 0$. Now consider the $n-1, 1$ entry. Using that $u_{n1} = 0$, we have $a_{n-1,n-1} u_{n-1,1} = u_{n-1,1} b_{11}$. The same argument now shows that $u_{n-1,1} = 0$. One continues in this way to show that $u_{i1} = 0$, $i > 1$. One then moves to the second column, and arguing in this way (one can make a more formal induction argument), one concludes that U is upper triangular. Since it is also unitary, it must be diagonal.

(d) This is a direct computation. We have

$$A^* A = \begin{pmatrix} \bar{\lambda}_1 & 0 \\ \bar{t}_{12} & \bar{\lambda}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & t_{12} \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} |\lambda_1|^2 & \bar{\lambda}_1 t_{12} \\ \bar{t}_{12} \lambda_1 & |t_{12}|^2 + |\lambda_2|^2 \end{pmatrix}$$

so that $\operatorname{tr} A^* A = |t_{12}|^2 + |\lambda_1|^2 + |\lambda_2|^2$.

(e) We proved in class that similar matrices have the same trace: *Given an* $n \times n$ *matrix* A*, we have that the characteristic polynomial is given as* $p_A(t) = t^n - \operatorname{tr} A t^{n-1} + \cdots + (-1)^n \det A$*. Since the characteristic polynomials of similar matrices are equal (since determinants of similar matrices are equal), this implies their traces are equal*. Now suppose that $B = U^* A U$ for some unitary matrix U. Then $\operatorname{tr}(B^* B) = \operatorname{tr}(U^* A^* U U^* A U) = \operatorname{tr}(U^* A^* A U) = \operatorname{tr}(A^* A)$.

10
10 points

10. **TRUE** or **FALSE**. You do **not** need to justify your answer.

10.(a). Let $(V, (-,-))$ be a Euclidean space, and let $v, w \in V$. Then

$$||v + w||^2 = ||v||^2 + ||w||^2$$

if and only if v and w are orthogonal.

T F FALSE: $||v+w||^2 = (v+w, v+w) = ||v||^2 + ||w||^2 + (v,w) + \overline{(v,w)} = ||v||^2 + ||w||^2 + 2\,\mathrm{Re}(v,w)$. So for instance, let $0 \neq v \in V$. Then $||v+iv||^2 = ||v||^2 + ||v||^2 + 2\,\mathrm{Re}(v,iv) = ||v||^2 + ||v||^2$, but $(v,iv) = -i(v,v) \neq 0$. (The statement of the problem would be true if we restricted ourselves to real Euclidean spaces.)

10.(b). Suppose that $T : V \to V'$ is a linear map of finite dimensional vector spaces. Then $\dim V' = \dim \ker(T) + \dim \mathrm{Im}(T)$.

T F FALSE: Take $V = \mathbb{R}$ and $V' = 0$. (The Rank–Nullity Theorem states that $\dim V = \dim \ker(T) + \dim \mathrm{Im}(T)$.)

10.(c). The cofactor matrix of an $n \times n$ matrix can only have rank equal to n, 1 or 0.

T F TRUE: We showed this in Midterm II.

10.(d). Suppose that $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ is symmetric, and let $v_1, v_2 \in \mathbb{R}^n$ be eigenvectors with corresponding eigenvalues λ_1, λ_2. If $\lambda_1 \neq \lambda_2$, then v_1 is orthogonal to v_2.

T F TRUE: $\lambda_1(v_1, v_2) = (Av_1, v_2) = (v_1, Av_2) = \lambda_2(v_1, v_2)$; since $\lambda_1 \neq \lambda_2$, we must have $(v_1, v_2) = 0$.

10.(e). The row space of a matrix is the same as the row space of the reduced row echelon form of the matrix.

T F TRUE: We have seen this in class. The RREF is obtained by taking a sequence of elementary row operations, and the elementary row operations preserve the row space.

10.(f). Suppose that M is an $n \times n$ matrix and $M^N = 0$ for some integer $N > 1$. Then M is diagonalizable.

T F FALSE: The matrix $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $M^2 = 0$, but M is not diagonalizable. (Note more generally that if $M = S^{-1}DS$ for a diagonal matrix D, then $0 = M^n = S^{-1}D^nS$ if and only if $D = 0$ (and hence $M = 0$), since S and S^{-1} induce isomorphisms.)

10.(g). Let A be an $n \times n$ matrix. Then $p_A(A) = 0$.

T F TRUE: This is the Cayley–Hamilton Theorem.

10.(h). Let $(V, (-,-))$ be a Euclidean space, and let $v, w \in V$. Then $|v.w| \leq ||v||||w||$.

T F TRUE: This is Cauchy–Schwarz.

10.(i). An $n \times n$ matrix has n linearly independent eigenvectors if and only if it has n distinct eigenvalues.

T F FALSE: Consider the identity matrix.

10.(j). Let $(V, (-, -))$ be a Euclidean space, and let $v, w \in V$. Then

$$||v + w|| \leq ||v|| + ||w||.$$

T F TRUE: This is the triangle inequality.