Solns: 1/29/18

Math 2300-007: Trig. Substitution

(Thanks to Faan Tone Liu)

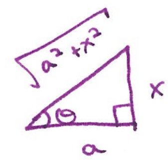
Key Points:

- Use these substitutions when you see integrals with $\sqrt{a^2 - x^2}$ or $\sqrt{a^2 + x^2}$.
- Substitution for $\sqrt{a^2 + x^2}$:

$x = a\tan\theta \qquad dx = a\sec^2\theta\, d\theta$

Method I

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2 + a^2\tan^2\theta} \\ &= \sqrt{a^2}\sqrt{1+\tan^2\theta} \\ &= |a|\sqrt{\sec^2\theta} \\ &= |a|\cdot|\sec\theta| = a\sec\theta \text{ if } a\sec\theta \geq 0\end{aligned}$$



$\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$

$\Updownarrow$

$\sqrt{a^2+x^2} = a\sec\theta$

$\tan\theta = \frac{x}{a}$

$x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

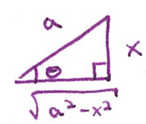
Method II

- Substitution for $\sqrt{a^2 - x^2}$:

$x = a\sin\theta \qquad dx = a\cos\theta\, d\theta$

Method I

$$\begin{aligned}\sqrt{a^2-x^2} &= \sqrt{a^2 - a^2\sin^2\theta} \\ &= \sqrt{a^2}\sqrt{1-\sin^2\theta} \\ &= |a|\sqrt{\cos^2\theta} \\ &= |a|\,|\cos\theta| \\ &= a\cos\theta \quad (\text{if } a, \cos\theta \geq 0)\end{aligned}$$



$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$\sqrt{a^2-x^2} = a\cos\theta$

- It's worth remembering:

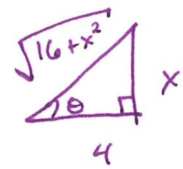
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

- Other notes and tips:

⚠ Here, we are assuming "a" and "$\cos\theta$" are non-negative so that $\sqrt{a^2} = |a| = a$ and $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$. If this is not the case (e.g. $\cos\theta$ is negative), we can replace $|\cos\theta|$ with $-\cos\theta$.

Compute the following integrals:

1. $\displaystyle\int \frac{1}{\sqrt{16+x^2}}\,dx = \int \frac{1}{4}\cos\theta \cdot 4\sec^2\theta \, d\theta$



$= \int \sec\theta \, d\theta$

$= \int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\, d\theta$

$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$

u-sub

$\left\{ \begin{array}{l} u = \sec\theta + \tan\theta \\ du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta \end{array} \right\}$ $= \int \frac{1}{u}\,du$

$= \ln|u| + C$

Use SOH CAHTOA and the triangle

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left| \frac{\sqrt{16+x^2}}{4} + \frac{x}{4} \right| + C$

$\cos\theta = \frac{4}{\sqrt{16+x^2}}$

Need This — $\left[\frac{1}{4}\cos\theta = \frac{1}{\sqrt{16+x^2}} \right]$

$\tan\theta = \frac{x}{4}$

$\left[x = 4\tan\theta \right]$

Need this — $\left[dx = 4\sec^2\theta \, d\theta \right]$

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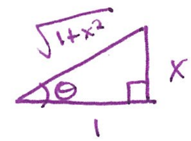
2. $\displaystyle\int x^3\sqrt{1+x^2}\,dx$



$= \int \tan^3\theta \cdot \sec\theta \cdot \sec^2\theta \, d\theta$

$= \int \tan^2\theta\sec^2\theta \cdot \underbrace{\sec\theta\tan\theta \, d\theta}$

$= \int (\sec^2\theta - 1)\sec^2\theta \cdot \overset{du}{\sec\theta\tan\theta \, d\theta}$ $\left\{ \begin{array}{l} u = \sec\theta \\ du = \sec\theta\tan\theta\,d\theta \end{array} \right\}$

$= \int (u^2-1)u^2\,du$

$= \int u^4 - u^2\,du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + C$

Use SOH CAHTOA and the triangle. $(\sec\theta = \sqrt{1+x^2})$

$= \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$

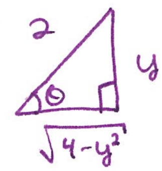
$\tan\theta = \frac{x}{1}$

Need these → $\left[\begin{array}{l} x = \tan\theta \\ dx = \sec^2\theta\,d\theta \\ \sec\theta = \sqrt{1+x^2} \end{array} \right.$

⚠ Note: This problem could be done with a u-sub:

$\left\{ \begin{array}{l} u = 1+x^2 \Leftrightarrow x^2 = u-1 \\ du = 2x\,dx \end{array} \right\}$

3. $\displaystyle\int \frac{1}{y^2\sqrt{4-y^2}}\,dy$



$$= \int \frac{1}{y^2}\cdot\frac{1}{\sqrt{4-y^2}}\,dy$$

$$= \int \frac{1}{4\sin^2\theta}\cdot\cancel{\tfrac{1}{2}}\cancel{\sec}\theta\cdot\cancel{2}\cancel{\cos}\theta\,d\theta$$

$$= \frac{1}{4}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

Use the triangle

$$= -\frac{1}{4}\,\frac{\sqrt{4-y^2}}{y} + C$$

$\sin\theta = \frac{y}{2}$

Need these

$y = 2\sin\theta$

$dy = 2\cos\theta\,d\theta$

$\sec\theta = \frac{2}{\sqrt{4-y^2}}$

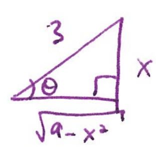
$\frac{1}{2}\sec\theta = \frac{1}{\sqrt{4-y^2}}$

4. $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$ (What other method could you use?)

u-sub

$u = 9-x^2 \Leftrightarrow x^2 = 9-u$

$du = -2x\,dx$



$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$\frac{1}{3}\sec\theta = \frac{1}{\sqrt{9-x^2}}$

Need these

$\sin\theta = \frac{x}{3}$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int 3^3\sin^3\theta\cdot\tfrac{1}{3}\cancel{\sec}\theta\cdot 3\cancel{\cos}\theta\,d\theta$$

$$= 27\int \sin^3\theta\,d\theta$$

$$= 27\int \sin^2\theta\cdot\sin\theta\,d\theta$$

$$= 27\int (1-\cos^2\theta)\cdot\sin\theta\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$= -27\int 1-u^2\,du$$

$$= -27\left[u - \tfrac{1}{3}u^3\right] + C$$

$$= -27\left[\cos\theta - \tfrac{1}{3}\cos^3\theta\right] + C$$

Use the triangle

$$= -27\left[\frac{\sqrt{9-x^2}}{3} - \frac{1}{3}\left(\sqrt{9-x^2}\right)^3\right] + C$$

5. $\displaystyle\int z\sqrt{1-z^2}\,dz$ (Hint: Can you do this another way?)

u-sub

$$\left\{\begin{array}{l} u = 1-z^2 \\ du = -2z\,dz \\ -\frac{1}{2}du = z\,dz \end{array}\right\}$$

$$\int z\sqrt{1-z^2}\,dz = -\frac{1}{2}\int\sqrt{u}\,du$$
$$= -\frac{1}{2}\int u^{1/2}\,du$$
$$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$
$$= -\frac{1}{3}(1-z^2)^{3/2} + C$$

6. $\displaystyle\int \frac{1}{36+x^2}\,dx$ (Hint: Try some sneaky algebra first.)

$$= \frac{1}{36}\int\frac{1}{1+\frac{x^2}{36}}\,dx$$
$$= \frac{1}{36}\int\frac{1}{1+(\frac{x}{6})^2}\,dx \qquad \left\{\begin{array}{l} u = \frac{x}{6} \\ du = \frac{1}{6}dx \\ 6\,du = dx \end{array}\right\}$$
$$= \frac{6}{36}\int\frac{1}{1+u^2}\,du$$
$$= \frac{1}{6}\arctan(u) + C$$
$$= \frac{1}{6}\arctan(\tfrac{x}{6}) + C$$

7. $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$ (Hint: Complete the square.)

$$= \int\frac{1}{x^2+2x+1-1+5}$$
$$= \int\frac{1}{(x+1)^2+4}\,dx \qquad \left\{\begin{array}{l} u = x+1 \\ du = dx \end{array}\right\}$$
$$= \int\frac{1}{u^2+4}\,du$$
$$= \frac{1}{2}\arctan(\tfrac{u}{2}) + C \qquad \text{(see \# 6 or tips on pg 1)}$$
$$= \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$$