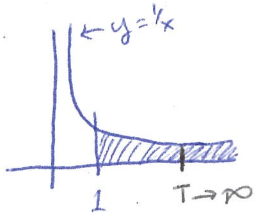
§5.10: Improper Integrals

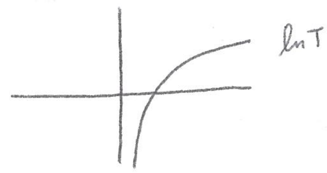
Solutions: 2/8/18

Decide whether or not the following improper integrals converge or diverge.

Type I: Integrals over infinite intervals

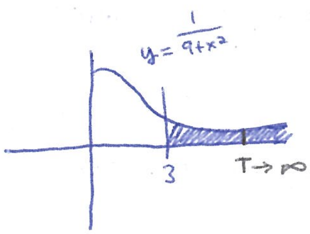
1. $\int_1^\infty \frac{1}{x}\,dx = \lim_{T\to\infty} \int_1^T \frac{1}{x}\,dx = \lim_{T\to\infty} \ln|x| \Big|_1^T = \lim_{T\to\infty} (\ln T - 0) = \infty$, so the integral diverges.

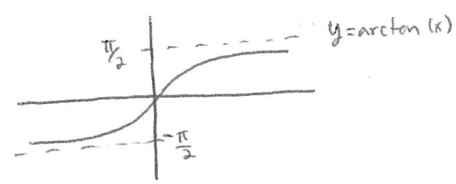




2. $\int_3^\infty \frac{1}{9+x^2}\,dx = \lim_{T\to\infty} \int_3^T \frac{1}{9+x^2}\,dx \overset{\text{see Trig. sub worksheet}}{=} \lim_{T\to\infty} \frac{1}{3}\arctan(x/3)\Big|_3^T = \lim_{T\to\infty} \left[\frac{\arctan(T/3)}{3} - \frac{\arctan(1)}{3}\right]$

$= \frac{\pi/2}{3} - \frac{\pi/4}{3} = \pi/6$, so the integral converges to $\pi/6$.





3. $\int_1^\infty \frac{1}{x^p}\,dx$, where $p \neq 1$ $= \lim_{T\to\infty} \int_1^T x^{-p}\,dx = \lim_{T\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^T = \lim_{T\to\infty} \left[\frac{T^{1-p}}{1-p} - \frac{1}{1-p}\right]$

$= \begin{cases} 0 - \frac{1}{1-p} & \text{if } 1-p<0 \\ \infty - \frac{1}{1-p} & \text{if } 1-p>0 \end{cases} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1. \end{cases}$

$\boxed{\text{p-test: (p-integrals)} \quad \int_1^\infty \frac{1}{x^p}\,dx \; \begin{cases} \text{Converges if } p>1 \\ \text{Diverges if } p\le 1 \end{cases}}$

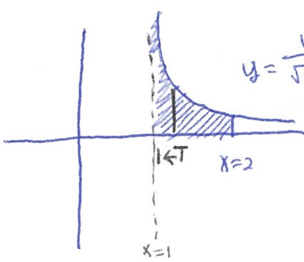
* We did the case $p=1$ in #1.

Type II: Integrals of functions with vertical asymptotes

Could do u-sub

4. $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{T\to 1^+} \int_T^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{T\to 1^+} \int_T^2 (x-1)^{-1/2}\,dx = \lim_{T\to 1^+} \left[2(x-1)^{1/2}\right]_T^2$

$\displaystyle = \lim_{T\to 1^+}\left[2\sqrt{1} - 2\sqrt{T-1}\right] = 2 - 2\sqrt{1-1} = 2$, so the integral converges. to 2.



* Improper because of vertical asymptote.

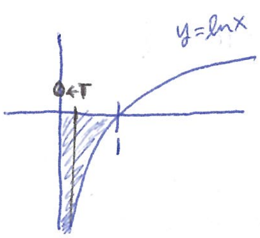
integration by parts $u = \ln x$, $dv = dx$

5. $\displaystyle\int_0^1 \ln(x)\,dx = \lim_{T\to 0^+} \int_T^1 \ln x\,dx = \lim_{T\to 0^+}\left[x\ln x - x\right]_T^1 = (1\ln 1 - 1) - \lim_{T\to 0^+}(T\ln T - T)$

form $(0\cdot\infty)$ (form: $\frac{\infty}{\infty}$) l'Hôpital's Rule

$\displaystyle = -1 + 0 - \lim_{T\to 0^+} T\ln T = -1 - \lim_{T\to 0^+} \frac{\ln T}{\frac{1}{T}} \overset{H}{=} -1 - \lim_{T\to 0^+} \frac{\frac{1}{T}}{\frac{-1}{T^2}}$

$\displaystyle = -1 - \lim_{T\to 0^+}(-T) = -1 - 0 = -1$, so the integral converges to -1.

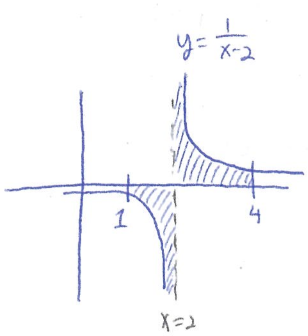


* Use l'Hôpital's Rule for limits of indeterminate form. Need to rearrange into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ <u>before</u> using l'Hôpital's Rule.

6. $\displaystyle\int_1^4 \frac{1}{x-2}\,dx = \int_1^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx = \lim_{T\to 2^-} \int_1^T \frac{1}{x-2}\,dx + \lim_{S\to 2^+} \int_S^4 \frac{1}{x-2}\,dx$

$\displaystyle = \lim_{T\to 2^-}\left[\ln|x-2|\right]_1^T + \lim_{S\to 2^+}\left[\ln|x-2|\right]_S^4 = \lim_{T\to 2^-}\left(\ln|T-2| - 0\right) + \lim_{S\to 2^+}\left(\ln 2 - \ln|S-2|\right)$

$= -\infty + \infty$, so the integral <u>diverges</u>. Note: $-\infty + \infty \neq 0$ because there is no way to compare the sizes of the infinities.



* We have to break this integral into two improper integrals and do each seperately. This is the <u>definition</u> of integrating over an asymptote.

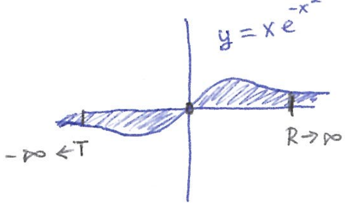
Miscellaneous

7. $$\int_{-\infty}^{\infty} te^{-t^2}\,dt = \int_{-\infty}^{0} te^{-t^2}dt + \int_{0}^{\infty} te^{-t^2}dt = \lim_{T\to-\infty}\int_{T}^{0} te^{-t^2}dt + \lim_{R\to\infty}\int_{0}^{R} te^{-t^2}dt$$

can do u-sub $\begin{bmatrix} u=-t^2 \\ du=-2t \end{bmatrix}$

$$= \lim_{T\to-\infty} -\frac{1}{2}e^{-t^2}\Big|_{T}^{0} + \lim_{R\to\infty}\left[-\frac{1}{2}e^{-t^2}\right]_0^R = \lim_{T\to-\infty}\left[\frac{-1}{2} + \underbrace{\cancel{\frac{1}{2e^{T^2}}}}_{\text{goes to 0}}\right] + \lim_{R\to\infty}\left[\underbrace{\cancel{\frac{-1}{2e^{R^2}}}}_{\text{goes to 0}} + \frac{1}{2}\right]$$

$$= \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0,$$ so this integral converges to 0.



* Can break this into two improper integrals around any point. I chose 0 because it seemed convenient.

* Need to break into two integrals by definition.

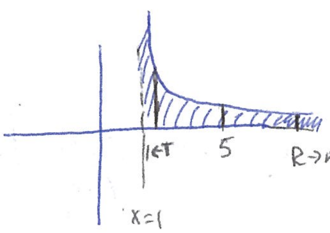
8. $$\int_{1}^{\infty} \frac{1}{x\ln(x)}\,dx = \int_1^5 \frac{1}{x\ln x}dx + \int_5^\infty \frac{1}{x\ln x}dx = \lim_{T\to1^+}\int_T^5 \frac{1}{x\ln x}dx + \lim_{R\to\infty}\int_5^R \frac{1}{x\ln x}dx$$

$\left\{ \begin{array}{l} \text{do u-sub} \\ u=\ln x \\ du=\frac{1}{x}dx \end{array} \right.$

$$= \lim_{T\to1^+}\Big[\ln|\ln x|\Big]_T^5 + \lim_{R\to\infty}\Big[\ln|\ln x|\Big]_5^R = \lim_{T\to1^+}\Big[\ln|\ln 5| - \ln|\ln T|\Big] + \lim_{R\to\infty}\Big[\ln|\ln R| - \ln|\ln(5)|\Big]$$

$= \infty + \infty$

$= \infty$, so the integral diverges

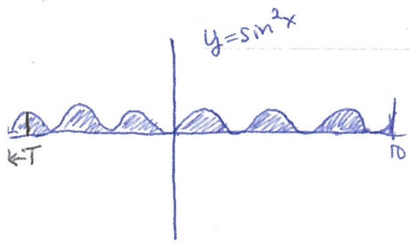


* I broke this one up around $x=5$. Any number between 1 and ∞ will do.

9. $$\int_{-\infty}^{10} \sin^2 x\,dx = \lim_{T\to-\infty}\int_T^{10} \sin^2 x\,dx = \lim_{T\to-\infty}\int_T^{10} \frac{1}{2}(1-\cos(2x))\,dx$$

$$= \lim_{T\to-\infty}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_T^{10} = \lim_{T\to-\infty}\left[\left(5 - \frac{1}{4}\sin(20)\right) - \left(\frac{T}{2} - \frac{1}{4}\sin(2T)\right)\right]$$

This limit does not exist, so the integral diverges.



Comparison Test

10. $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$

Compare to $\displaystyle\int_3^\infty \frac{1}{\sqrt{x}}\,dx$, which is smaller and diverges by p-test $(p=\tfrac{1}{2}\leq 1)$

Hypotheses: $\frac{\ln x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \geq 0$ on $[3,\infty)$, and both functions are continuous on this interval.

Conclusion: Since $0 \leq \frac{1}{\sqrt{x}} \leq \frac{\ln x}{\sqrt{x}}$, we have $0 \leq \displaystyle\int_3^\infty \frac{1}{\sqrt{x}}\,dx \leq \int_3^\infty \frac{\ln x}{\sqrt{x}}\,dx$.
Now, the smaller integral, $\displaystyle\int_3^\infty \frac{1}{\sqrt{x}}\,dx$ diverges because it is a p-integral with $p=\tfrac{1}{2}\leq 1$. It follows that the larger integral $\displaystyle\int_3^\infty \frac{\ln x}{\sqrt{x}}\,dx$ diverges, too by the comparison Theorem.

11. $\displaystyle\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx$

Compare to $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx$, which is larger and $\boxed{\text{convergent}}$. (see #2 on pg 1)

Hypotheses: $0 \leq \frac{|\sin x|}{x^2+1} \leq \frac{1}{x^2+1}$ on $[1,\infty)$, and both functions are continuous on this interval.

Conclusion: Since $0 \leq \frac{|\sin x|}{x^2+1} \leq \frac{1}{x^2+1}$ on $[1,\infty)$, we have $0 \leq \displaystyle\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx \leq \int_1^\infty \frac{1}{x^2+1}\,dx$.
The larger integral, $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx$ converges $\left[\displaystyle\lim_{T\to\infty}\int_1^T \frac{1}{x^2+1}\,dx = \lim_{T\to\infty}(\arctan T - \arctan 1) = \tfrac{\pi}{2}-\tfrac{\pi}{4} = \tfrac{\pi}{4}\right.$
so the smaller integral, $\displaystyle\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx$ converges also, by the Comparison Theorem.