§6.6 Part II: Center of Mass

Solutions : 2/19/18

Key Points:

- The **center of mass** (or centroid) of a thin plate is:

  The balance point of the object 

- For a system of n particles with masses $m_1, \ldots, m_n$ located at the points $(x_1, y_1), \ldots, (x_n, y_n)$ in the xy-plane, the center of mass of the system is located at: $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{x_1 m_1 + \cdots + x_n m_n}{m_1 + \cdots + m_2}, \qquad \bar{Y} = \frac{y_1 m_1 + \cdots + y_n m_n}{m_1 + \cdots + m_n}$$

Notice that M_y uses the x-coords. and M_x uses the y-coords!

- The **moment of the system about the y-axis** is

$$M_y = x_1 m_1 + \cdots + x_n m_n = \sum_{i=1}^{n} x_i m_i$$

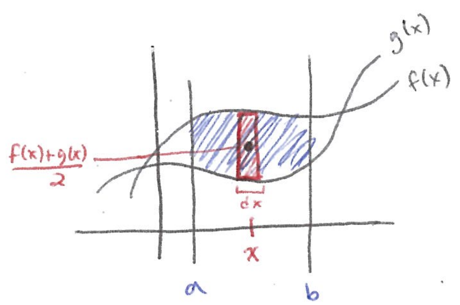
This measures The tendency of the system to rotate about the y-axis.

- The **moment of the system about the x-axis** is

$$M_x = y_1 m_1 + \cdots + y_n m_n = \sum_{i=1}^{n} y_i m_i$$

This measures the tendency of the system to rotate about the x-axis.

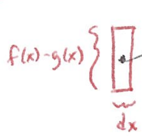
- In the case where we are looking at a thin region bounded by the curves $y = f(x)$ and $y = g(x)$, we chop the region in to small rectangles that we consider to be point masses. In this case:



$$\bar{X} = \frac{M_y}{\text{mass}} = \frac{\int_a^b \tilde{x} \cdot \text{mass}}{\rho \int_a^b f(x) - g(x)\,dx} = \frac{\cancel{\rho} \int_a^b x\,(f(x)-g(x))\,dx}{\cancel{\rho} \int_a^b f(x) - g(x)\,dx}$$

$$\boxed{\bar{X} = \frac{\int_a^b x\,(f(x)-g(x))\,dx}{\int_a^b f(x) - g(x)\,dx}}$$

$$\bar{Y} = \frac{M_x}{\text{mass}} = \frac{\int_a^b \tilde{y} \cdot \text{mass}}{\rho \int_a^b f(x) - g(x)\,dx} = \frac{\cancel{\rho} \int_a^b \frac{f(x)+g(x)}{2}\,(f(x)-g(x))}{\cancel{\rho} \int_a^b f(x) - g(x)\,dx}$$



center of slice : $(\tilde{x}, \tilde{y})$

$\tilde{x} = x$

$\tilde{y} = \frac{f(x)+g(x)}{2}$ (avg. of heights)

mass = Area $\cdot\, \rho$ = $\rho \cdot [f(x) - g(x)]\,dx$

mass density of region

$$\boxed{\bar{Y} = \frac{\frac{1}{2}\int_a^b [f(x)+g(x)] \cdot [f(x) - g(x)]\,dx}{\int_a^b f(x) - g(x)\,dx}}$$

Examples:

1. Find the moments M_x and M_y and the center of mass of the system of the following point masses:

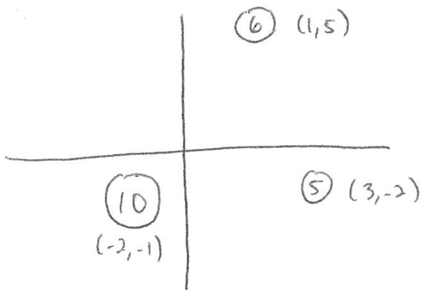
   - A mass of 6 at the point (1,5)
   - A mass of 5 at the point (3,-2)
   - A mass of 10 at the point (-2,-1)

y-coords

$$M_x = 6 \cdot 5 + 5(-2) + 10(-1) = 10$$

$$M_y = 6 \cdot 1 + 5 \cdot 3 + 10(-2) = 1$$

x-coords

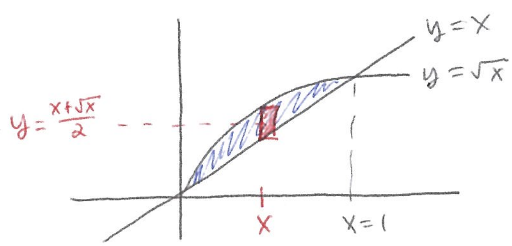


$$\bar{X} = \frac{M_y}{\text{total mass}} = \frac{1}{6+5+10} = \frac{1}{21}$$

$$\bar{Y} = \frac{M_x}{\text{total mass}} = \frac{10}{6+5+10} = \frac{10}{21}$$

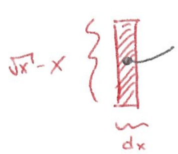
$$\boxed{\begin{array}{l} \text{COM}: \left(\frac{1}{21}, \frac{10}{21}\right) \\ M_x: 10 \\ M_y: 1 \end{array}}$$

2. Find the centroid of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.



$$\bar{X} = \frac{\int_0^1 \tilde{x} \cdot \text{mass}}{\int_0^1 \text{mass}} = \frac{\int_0^1 x(\sqrt{x}-x)\,dx}{\int_0^1 \sqrt{x}-x\,dx} = \cdots = \frac{1/15}{1/6}$$

$$= \frac{6}{15} = \frac{2}{5}$$



center $(\tilde{x}, \tilde{y})$

$\tilde{x} = x$

$\tilde{y} = \frac{x+\sqrt{x}}{2}$

$\text{mass} = \rho \cdot \text{Area}$

$= 1 \cdot (\sqrt{x}-x)\,dx$

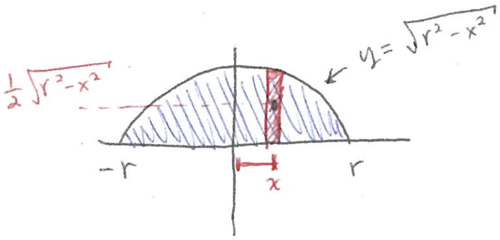
(Assume $\rho = 1$. see #3 for explanation.)

$$\bar{Y} = \frac{\int_0^1 \tilde{y} \cdot \text{mass}}{\int_0^1 \text{mass}} = \frac{\int_0^1 \frac{x+\sqrt{x}}{2} \cdot (\sqrt{x}-x)\,dx}{\int_0^1 \sqrt{x}-x\,dx} = \cdots = \frac{1/12}{1/6}$$

$$= \frac{6}{12} = \frac{1}{2}$$

$$\boxed{\text{COM}: \left(\tfrac{2}{5}, \tfrac{1}{2}\right)}$$

3. Find the center of mass of the semicircular plate of radius r.



[Assume $\rho = 1$ because ρ "cancels" out in COM calculation. Note: Need ρ for M_x or M_y, but we're not looking for that here]

$$\bar{X} = \frac{\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx}{\int_{-r}^{r}\sqrt{r^2-x^2}\,dx} \leftarrow \frac{\pi r^2}{2} \text{ Area of semi-circle}$$

$$= \frac{2}{\pi r^2}\cdot\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx \qquad \left\{\begin{array}{l} u = r^2-x^2 \\ du = -2x\,dx \end{array}\right\}$$

$$= \frac{2}{\pi r^2}\cdot\frac{-1}{2}\int_{u=0}^{u=0}\sqrt{u}\,du$$

$$= \frac{-1}{\pi r^2}\cdot 0 = 0$$

$\frac{1}{2}\sqrt{r^2-x^2}$ center $(\tilde{x},\tilde{y})$

$\tilde{x} = x$

$\tilde{y} = \frac{1}{2}\sqrt{r^2-x^2}$

mass $= \sqrt{r^2-x^2}\,dx\cdot\rho$

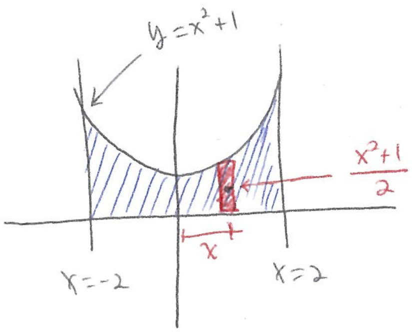
$= \sqrt{r^2-x^2}\,dx$

$$\bar{Y} = \frac{\int_{-r}^{r}\frac{1}{2}\sqrt{r^2-x^2}\cdot\sqrt{r^2-x^2}\,dx}{\int_{-r}^{r}\sqrt{r^2-x^2}\,dx} = \frac{\int_{-r}^{r}\frac{1}{2}(r^2-x^2)\,dx}{\frac{\pi r^2}{2}}$$

$$= \frac{\left[\frac{1}{2}r^2x - \frac{1}{6}x^3\right]_{-r}^{r}}{\frac{\pi r^2}{2}} = \frac{\left(\frac{1}{2}r^3-\frac{1}{6}r^3\right)-\left(-\frac{1}{2}r^3+\frac{1}{6}r^3\right)}{\frac{\pi r^2}{2}} = \frac{\frac{2}{3}r^3}{\frac{\pi r^2}{2}} = \frac{4r^3}{3\pi r^2}$$

$$\boxed{\text{COM: } \left(0, \frac{4}{3\pi}r\right)}$$

4. Find the center of mass of the region between the x-axis and the parabola $y = x^2+1$ between $x = -2$ and $x = 2$.



$$\bar{X} = \frac{\int_{-2}^{2} x(x^2+1)\,dx}{\int_{-2}^{2}(x^2+1)\,dx} = 0$$

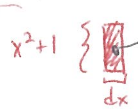
by symmetry of the region (or can integrate to double check)

$$\bar{Y} = \frac{\int_{-2}^{2}\frac{x^2+1}{2}\cdot(x^2+1)\,dx}{\int_{-2}^{2}(x^2+1)\,dx}$$

$$= \frac{\frac{1}{2}\int_{-2}^{2}x^4+2x^2+1\,dx}{\left[\frac{1}{3}x^3+x\right]_{-2}^{2}}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{5}x^5+\frac{2}{3}x^3+x\right)_{-2}^{2}}{\frac{28}{3}}$$

$$= \frac{\frac{206}{15}}{\frac{28}{3}} = \frac{103}{70} \approx 1.471$$



x^2+1 center $(\tilde{x},\tilde{y})$

$\tilde{x} = x$

$\tilde{y} = \frac{x^2+1}{2}$

mass $= (x^2+1)\,dx\cdot\rho$

$= (x^2+1)\,dx$

Assume $\rho = 1$ for similar reasons to #3

[COM: $(0, 1.471)$

Notice that this point is outside the region. This sometimes happens when the region has a non-convex shape]