Solutions 3/10/18

§8.4: Part I - Alternating Series

(Thanks to Faan Tone Liu)

Key Points:

- If the terms in a series alternate signs, we call the series an **alternating series.**
- An alternating series can be written in the form

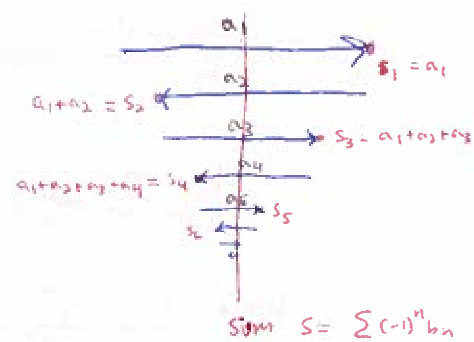
$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}b_n, \quad \text{where } b_n \geq 0.$$

  (i.e. b_n includes no negative terms)

[Consider the following Picture:]

- **Alternating series test:**

If ① b_n is decreasing and ② $\lim_{n\to\infty} b_n = 0$, Then, the alternating series $\sum(-1)^n b_n$ converges.



- Note: Recall that to show b_n is decreasing, show

  $a_{n+1} > a_n$ or $\frac{a_{n+1}}{a_n} < 1$ or $f'(x) < 0$

  ↑ for "connect the dots" function $f(x)$ [i.e. $f(n) = a_n$]

- Note: If in an alternating series, $\lim_{n\to\infty} b_n \neq 0$, then

  $\lim_{n\to\infty}(-1)^n b_n$ does not exist, so $\sum(-1)^n b_n$ diverges by the divergence test.

- Alternating series remainder test: If $\sum(-1)^n b_n$ converges by the alternating series test, then

$$|\text{Error}| = |R_n| = |S - S_n| \leq b_{n+1}$$

  ↑ $|a_{n+1}|$

  i.e. error given by the size of the $(n+1)$st term.

Examples:

1. (Alternating Harmonic Series) Does $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converge or diverge?

$b_n = \frac{1}{n}$

① $b_n = \frac{1}{n}$ decreasing because $\frac{1}{n+1} < \frac{1}{n}$.

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$

By Alternating series test, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges.

2. Does $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converge or diverge?

$b_n = \frac{\ln n}{n}$

① $f(x) = \frac{\ln x}{x}$. $f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$ for $x \geq e$

So b_n decreasing for $n \geq 3$.

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0$

By alternating series test, the series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges.

3. Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{e^n}$ converge or diverge?

$\lim_{n\to\infty} \frac{n!}{e^n} = \lim_{n\to\infty} \frac{1}{e} \cdot \frac{2}{e} \cdot \underbrace{\frac{3}{e} \cdots \frac{n}{e}}_{\geq 3/e} \geq \lim_{n\to\infty} \frac{1}{e} \cdot \frac{2}{e} \cdot \left(\frac{3}{e}\right)^{n-2} = \infty$.

Hence $\lim_{n\to\infty} \frac{(-1)^n n!}{e^n}$ does not exist. By the divergence test, the series $\sum \frac{(-1)^n n!}{e^n}$ diverges.

4. Estimate $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^5}$ using three terms. How accurate is your estimate?

$S_3 = \sum_{n=1}^{3} (-1)^{n+1} \frac{1}{n^5} = \frac{1}{1} - \frac{1}{2^5} + \frac{1}{3^5} = \frac{7565}{7776} \approx 0.973$

$|\text{error}| = |s - 0.973| \leq b_{3+1} = \frac{1}{4^5} \approx 0.000977$

This means that the true value of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^5}$ is within 0.000977 of our guess of 0.973.

5. How many terms should we add to ensure that our estimate of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is within 0.0001 of the true value?

We want $|\text{error}| \leq 0.0001$. We know $b_{n+1} = \frac{1}{\sqrt{n+1}} \geq |\text{error}|$.

Hence, we should solve for n to make the following true:

$0.0001 \geq \frac{1}{\sqrt{n+1}} \geq |\text{error}|$

$\sqrt{n+1} \geq \frac{1}{0.0001}$

$n+1 \geq \left(\frac{1}{0.0001}\right)^2 \rightarrow n \geq (10000)^2 - 1$

$\boxed{n \geq 10^8 - 1}$

We need to add a lot of terms because $\frac{1}{\sqrt{n}}$ goes to zero relatively slowly.