Solutions

Goal: If we know that a power series converges to a specific function, we can manipulate the equation to determine the limits of new power series. This is a nifty and fast way to get lots of new power series representations of functions. Today we will manipulate power series in these ways:

- Substitute
- Multiply by x
- Differentiate
- Integrate

1. Write down a power series representation for the function $f(x) = \frac{1}{1-x}$ by using the fact that the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

$$f(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

The series converges to $\frac{1}{1-x}$ on the interval $-1 < x < 1$.

2. Using your response for the last problem, substituting $-x$ in the place of x, find the power series representation for $f(x) = \frac{1}{1+x}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

Interval of convergence: $-1 < -x < 1 \Rightarrow 1 > x > -1$, so $(-1, 1)$

3. Find the power series representation for $f(x) = \frac{1}{1+x^2}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

[Hint: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$]

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Interval of convergence: $-1 < -x^2 < 1 \Rightarrow 1 > x^2 > -1 \Rightarrow -1 < x < 1$

* Need to check endpoints if we did differentiation or integration to the power series

4. Find the power series representation for $\frac{x}{1-x}$. (Hint: multiply answer to problem 1 by x.) On what interval does the series converge to the function?

$$\frac{x}{1-x} = x \cdot \frac{1}{1-x} = x(1+x+x^2+\cdots) = x+x^2+x^3+\cdots = \sum_{n=1}^{\infty} x^n$$

(true for x in (-1,1) because that's when we can say $\frac{1}{1-x} = 1+x+x^2+\cdots$)

IOC: $(-1,1)$

5. Find the power series representation for $\frac{1}{(1-x)^2}$. On what interval does the series converge to the function? Hint: Take the derivative of both sides of this equation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n$$

This is true for x in (-1,1), so we want to say that the IOC is (-1,1), but we differentiated, so we need to check the endpoints:

$$\frac{d}{dx}\left[(1-x)^{-1}\right] = 0 + 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \sum_{n=0}^{\infty} nx^{n-1}$$

$$-(1-x)^{-2}\cdot(-1) = 1+2x+3x^2+\cdots+nx^{n-1}+\cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\boxed{\frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots = \sum_{n=1}^{\infty} nx^{n-1}}$$

$x=-1$: $\sum_{n=1}^{\infty} n(-1)^{n-1}$ diverges by div. test

$x=1$: $\sum_{n=1}^{\infty} n$ diverges by div. test.

$\boxed{\text{IOC: } (-1,1)}$

6. Find the power series representation of $\arctan x$. (Hint: start with the power series for $\frac{1}{1+x^2}$ and antidifferentiate. Solve for the constant of integration by substituting $x = 0$.) On what interval does the series converge to the function?

We know that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$, so

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + \cdots \,dx$$

This equality is true for x in (-1,1) by prob #3. Since we integrated, we need to check the endpoints:

$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots + C$$

Since $\arctan(0) = 0$, it follows that $C = 0$. Hence,

$$\boxed{\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$$

$x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n(-1)}{2n+1}$ Converges by AST

$x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1}{2n+1}$ converges by AST

$\boxed{\text{IOC: } [-1,1]}$