Math 2300-013: Trig. Substitution

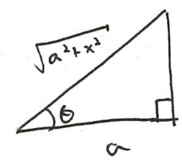
Key Points:

- Use these substitutions when you see integrals with $\sqrt{a^2 - x^2}$ or $\sqrt{a^2 + x^2}$.
- Substitution for $\sqrt{a^2 + x^2}$:

$X = a\tan\theta$

optional

$$\sqrt{a^2+x^2} = \sqrt{a^2 + (a\tan\theta)^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2}\sqrt{1+\tan^2\theta} = a\sqrt{\sec^2\theta} = a\sec\theta$$



$\tan\theta = \frac{x}{a}$

$X = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$

$\sqrt{a^2+x^2} = \frac{a}{\cos\theta} = \sec\theta$

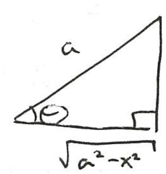
I like this method best because it's visual!

- Substitution for $\sqrt{a^2 - x^2}$:

$X = a\sin\theta$

optional

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2}\sqrt{1-\sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta$$



$\sin\theta = \frac{x}{a}$

$X = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$\sqrt{a^2-x^2} = a\cos\theta$

- It's worth remembering:

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

- Other notes and tips:

⚠ "a" is assumed to be positive in these (so $\sqrt{a^2} = |a| = a$)

⚠ $\cos\theta$ is assumed to be positive for similar reasons (we can adjust if $\cos\theta$ is negative)

Compute the following integrals:

1. $\displaystyle\int \frac{1}{\sqrt{16+x^2}}\,dx = \int \frac{1}{4}\cos\theta \cdot 4\sec^2\theta\, d\theta$

$= \frac{4}{4}\int \sec\theta\, d\theta$

$= \int \sec\theta\, d\theta$

$= \int \sec\theta \left(\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\right) d\theta$

$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$

$= \int \frac{1}{u}\, du$

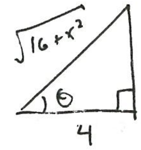
$= \ln|\sec\theta+\tan\theta| + C$

$= \ln\left|\frac{\sqrt{16+x^2}}{4} + \frac{x}{4}\right| + C$

u-sub:

$u = \sec\theta + \tan\theta$

$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta$



$\cos\theta = \frac{4}{\sqrt{16+x^2}}$ $\tan\theta = \frac{x}{4}$

Need these

$\frac{1}{4}\cos\theta = \frac{1}{\sqrt{16+x^2}}$

$x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

use the triangle

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2. $\displaystyle\int x^3\sqrt{1+x^2}\,dx$

$= \int (\tan\theta)^3 \cdot \sec\theta \cdot \sec^2\theta\, d\theta$

$= \int \tan^2\theta \cdot \sec^2\theta \cdot \sec\theta\tan\theta\, d\theta$

$= \int (\sec^2\theta - 1)\cdot \sec^2\theta \cdot \sec\theta\tan\theta\, d\theta$

$= \int (u^2-1)u^2\, du$

$= \int u^4 - u^2\, du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + C$

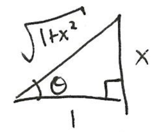
$= \frac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + C$

use triangle.

u-sub:

$u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$



Need these

$\tan\theta = x$

$\sec^2\theta\, d\theta = dx$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

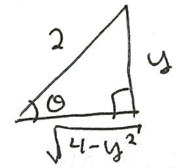
$\sec\theta = \sqrt{1+x^2}$

⚠ Note: This problem could be done with a u-sub:

$u = 1+x^2 \Rightarrow x^2 = u-$

$du = 2x\,dx$

3. $\displaystyle\int \frac{1}{y^2\sqrt{4-y^2}}\,dy$



$$= \int \frac{1}{4\sin^2\theta}\cdot\frac{1}{2}\sec\theta\cdot 2\cos\theta\, d\theta$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{4-y^2}}{y} + C$$

use the triangle

$\sin\theta = \frac{y}{2}$

Need these

$y = 2\sin\theta$

$dy = 2\cos\theta\, d\theta$

$\sec\theta = \frac{2}{\sqrt{4-y^2}}$

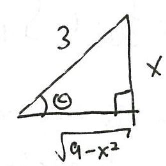
$\frac{1}{2}\sec\theta = \frac{1}{\sqrt{4-y^2}}$

4. $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$ (What other method could you use?)

u-sub

$u = 9 - x^2 \Rightarrow x^2 = 9 - u$

$du = -2x\,dx$



$$= \int \frac{(3\sin\theta)^3}{3\cos\theta}\cdot 3\cos\theta\, d\theta$$

$\sin\theta = \frac{x}{3}$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$3\cos\theta = \sqrt{9-x^2}$

$$= 3\int \sin^3\theta\, d\theta$$

$$= 3\int \sin^2\theta\cdot\sin\theta\, d\theta$$

$$= 3\int (1-\cos^2\theta)\cdot\sin\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$-du = \sin\theta\, d\theta$

$$= -3\int (1-u^2)\, du$$

$$= -3\left(u - \frac{1}{3}u^3\right) + C$$

$$= -3\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= -3\left[\frac{\sqrt{9-x^2}}{3} - \frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3\right] + C$$

5. $\int z\sqrt{1-z^2}\,dz$ (Hint: Can you do this another way?)

Use u-sub!

$$\left[\begin{array}{c} u = 1-z^2 \\ du = -2z\,dz \\ -\tfrac{1}{2}du = z\,dz \end{array}\right]$$

$= \frac{-1}{2}\int \sqrt{u}\,du$

$= -\frac{1}{2}\int u^{1/2}\,du$

$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$= -\frac{1}{3}(1-z^2)^{3/2} + C$

6. $\int \frac{1}{36+x^2}\,dx$ (Hint: Try some sneaky algebra first.)

$= \int \frac{1}{36(1+\frac{x^2}{36})}\,dx$

$= \frac{1}{36}\int \frac{1}{1+(\frac{x}{6})^2}\,dx$

$$\left[\begin{array}{c} u = \frac{x}{6} \\ du = \frac{1}{6}dx \\ 6\,du = dx \end{array}\right]$$

$= \frac{1}{36}\cdot 6\int \frac{1}{1+u^2}\,du$

$= \frac{1}{6}\arctan(u) + C$

$= \frac{1}{6}\arctan(\frac{x}{6}) + C$

7. $\int \frac{1}{x^2+2x+5}\,dx$ (Hint: Complete the square.)

$= \int \frac{1}{x^2+2x+\underline{1}-\underline{1}+5}\,dx$

$= \int \frac{1}{(x+1)^2+4}\,dx$

$$\left[\begin{array}{c} u = x+1 \\ du = dx \end{array}\right]$$

$= \int \frac{1}{u^2+4}\,du$

see tips on front and #6

$= \frac{1}{2}\arctan(\frac{u}{2}) + C$

$= \frac{1}{2}\arctan(\frac{x+1}{2}) + C$